A Remark on Partitions and Triangles

By

Takuya Doi and Shin-ichi Katayama

Department of Mathematical Sciences,
Faculty of Integrated Arts and Sciences
The University of Tokushima, Tokushima 770-8502, JAPAN

e-mail address : *c100941033@stud.tokushima-u.ac.jp*
: *katayama@ias.tokushima-u.ac.jp*
(Received September 30, 2009)

Abstract

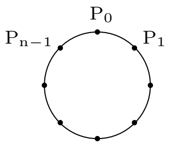
Let C be a circle divided into n parts equally. $S = \{P_0, P_1, \ldots, P_{n-1}\}$ denotes the set of the ends of these parts on C. Let $C_3(n)$ be the number of incongruent triangles inscribed in C, where the vertices of the triangles are chosen from S. In this note, we shall show a relation between the number $C_3(n)$ and the partitions into at most three parts.

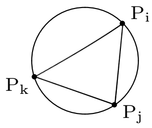
2000 Mathematics Subject Classification. Primary 05A17; Secondary 11P81

Introduction

Let C be a circle divided into n parts equally and the ends of parts be labeled $\{P_0, P_1, \ldots, P_{n-1}\}$ as in the following Figure 1:

Figure 1 **Figure 2**





Let $C_3(n)$ be the number of incongruent triangles $\triangle P_iP_jP_k$ with vertices P_i, P_j, P_k chosen from $S = \{P_0, P_1, \ldots, P_{n-1}\}$ as in Figure 2. In this note, we shall show a relation between the numbers $C_3(n)$ and the partitions into at most three parts. In the next section, we shall recall the fundamental notations and the terminology in [3].

Main Result

Let $p(n,m)$ be the number of partitions of n with each part $\leq m$, that is,

$$p(n,m) = p(n \mid \text{parts in } \{1,2,\ldots,m\}).$$

It is easy to show $p(n,1) = 1$ and $p(n,2) = \left[\frac{n}{2}\right] + 1$, where $[x]$ denotes the greatest integer $\leq x$. Then one can easily verify that $p(n,m)$ has the following generating function:

$$\sum_{n=0}^{\infty} p(n,m)q^n = \frac{1}{(1-q)(1-q^2)\cdots(1-q^m)}.$$

In the case $m = 3$, it has been shown in [3] that

$$p(n,3) = \left\{\frac{(n+3)^2}{12}\right\},$$

where $\{x\}$ denotes the nearest integer to x. Let $\triangle_n$ be the number of incongruent triangles with integer sides and perimeter n. This number $\triangle_n$ has been investigated in [1] and [2]. In [1] and [3], it has been proved that

$$\triangle_n = p(n-3 \mid \text{parts in } \{2,3,4\}) = \left\{\frac{n^2}{12}\right\} - \left[\frac{n}{4}\right]\left[\frac{n+2}{4}\right].$$

Thus the generating function of $\triangle_n$ is given by

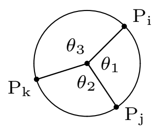
$$\sum_{n=0}^{\infty} \triangle_n q^n = \frac{q^3}{(1-q^2)(1-q^3)(1-q^4)}.$$

In the following, we shall show the similar results also hold for $C_3(n)$. Let O be the center of the circle C. Let P_i, P_j, P_k $(0 \leq i < j < k \leq n-1)$ be the vertices chosen from $S = \{P_0, P_1, \ldots, P_{n-1}\}$. We denote the central angles of the inscribed triangle $\triangle P_i P_j P_k$ by

$$\angle P_i O P_j = \theta_1, \angle P_j O P_k = \theta_2, \angle P_k O P_i = \theta_3,$$

respectively as in the following Figure 3.

Figure 3



From the definition, we have $\theta_1 + \theta_2 + \theta_3 = 2\pi$. Put $a = j - i$, $b = k - j$ and $c = n + i - k$. Then we have

$$\theta_1 = \frac{2a\pi}{n}, \theta_2 = \frac{2b\pi}{n}, \theta_3 = \frac{2c\pi}{n}.$$

Then a, b, c are the natural numbers which satisfy the condition

$$a, b, c \geq 1 \quad \text{and} \quad a + b + c = n.$$

We note any inscribed triangle whose vertices $\{P_i, P_j, P_k\}$ in S is congruent to exactly one inscribed triangle whose central angles satisfy $n = a + b + c$ and $a \geq b \geq c \geq 1$. Thus we have shown

$$\begin{aligned} C_3(n) &= p(n \mid n = a + b + c, \text{with } a \geq b \geq c \geq 1 \ ) \\ &= p(n - 3 | \ \text{ at most three parts}). \end{aligned}$$

The conjugation of Ferrers-Young diagram implies that

$$p(n - 3 \mid \ \text{ at most three parts}) = p(n - 3 \mid \text{parts in } \{1, 2, 3\}) = p(n - 3, 3).$$

Therefore we have shown the following theorem which is similar to the formula of $\triangle_n$.

Theorem. *With the above notation,*

$$C_3(n) = p(n - 3, 3) = \left\{ \frac{n^2}{12} \right\},$$

and the generating function of $C_3(n)$ is given by

$$\sum_{n=0}^{\infty} C_3(n) q^n = \frac{q^3}{(1 - q)(1 - q^2)(1 - q^3)}.$$

Corollary. *$R_3(n)$ denotes the number of incongruent right triangles where the vertices P_i, P_j, P_k are chosen from S. Then $R_3(n) > 0$ if and only if n is even and ≥ 4. Put $n = 2m$. Then we have*

$$R_3(n) = \left[\frac{m}{2}\right],$$

and the generating function of $R_3(n)$ is given by

$$\sum_{n=0}^{\infty} R_3(n)q^n = \sum_{m=0}^{\infty} \left[\frac{m}{2}\right] q^{2m} = \frac{q^4}{(1-q^2)(1-q^4)}.$$

Proof. We know the triangle $P_iP_jP_k$ is a righit triangle if and only if the largest a satisfies the condition $a = b + c$. Hence we have shown the corresponding central angles must satisfy $n = 2a$ and $a > b \geq c \geq 1$ with $b + c = a$. Put $a = m$. Then we have $n = 2m$ and $R_3(n) = \left[\frac{m}{2}\right]$, which completes the proof of this corollary.

Let us consider similar problems. Take two points $P_i, P_j \in S$ and consider the length of the line segment P_iP_j. In the same way as above, the small one of the central angles $\angle P_iOP_j$ can be expressed as $\frac{2a\pi}{n}$. Put $\frac{2b\pi}{n} = 2\pi - \frac{2a\pi}{n}$. Then a satisfies $1 \leq a \leq \left[\frac{n}{2}\right]$. We denote the number of line segments P_iP_j $(0 \leq i < j \leq n-1)$ with $P_i, P_j \in S$ of different length by $C_2(n)$. Then we have

$$C_2(n) = p(n \mid n = a + b \text{ with } b \geq a \geq 1) = p(n-2, 2) = \left[\frac{n}{2}\right].$$

Thus the generating function of $C_2(n)$ is given by

$$\sum_{n=0}^{\infty} C_2(n)q^n = \frac{q^2}{(1-q)(1-q^2)}.$$

Let $S_2(n)$ be the set of all the sides and the diagonals of a regular n-sided polygon. We note the number $C_2(n)$ coincides the number of segments $\in S_2(n)$ with the different length.

We may consider one more case $C_1(n)$. Let us choose the point $P_i \in S = \{P_0, P_1, \ldots, P_{n-1}\}$. Since all the points P_i congruent each other by the action of the congruent transformation group of the plane. Thus we may consider $C_1(n) = 1$ for $n \geq 1$, that is $C_1(n) = p(n-1, 1)$. Hence the generating function of $C_1(n)$ is given by

$$\sum_{n=0}^{\infty} C_1(n)q^n = \frac{q}{(1-q)}.$$

It shall be natural to consider the similar problems for $C_k(n)$ for $k \geq 4$. But the cases $k \geq 4$ seem to be more complicated than the cases $1 \leq k \leq 3$. For example consider the case $k = 4$, that is, $C_4(n)$ (the number of incongruent quadrilaterals $P_iP_jP_kP_l$, where $P_i, P_j, P_k, P_l \in S$). Then we see $C_4(n)$ is not equal to $p(n-4, 4)$. To understand this fact, it suffices to calculate $p(n-4, 4)$ and $C_4(n)$ for small values n as follows.

n	4	5	6	7	8	9	10
$p(n-4, n)$	1	1	2	3	5	6	9
$C_4(n)$	1	1	3	4	8	10	16

We hope our investigation on $C_4(n)$ will be published in the near future.

Remarks on $p(n \mid \text{parts in } \{2, 3, 4\})$

The investigation of this note started when we have encountered with the exercises 84 and 85 in the very interesting text book [3], where the formulas for $p(n \mid \text{parts in}\{2, 3, 4\})$ and $\triangle_n$ are given. Concerning these exercises, we have asked some question to Prof. G. E. Andrews and Prof. K. Eriksson who are the authors of the text [3]. They have kindly informed us the paper [1] is the origin of the exercise 85 and also informed us the survey paper [2] on these subjects. We have found that a direct proof of the formula of $p(n \mid \text{parts in } \{2, 3, 4\})$. It is easy to show the formula but has not been written explicitly in the papers [1], [2] and the text [3]. Thus hoping to be of some meaning to give a direct proof of those formulas, we shall write down the direct proofs of the exercises 84 and 85 in the rest of this note.

Firstly, we have the partial fractions decomposition:

$$\sum_{n=0}^{\infty} p(n \mid \text{parts in } \{2, 3, 4\}) q^n = \frac{1}{(1-q^2)(1-q^3)(1-q^4)}$$

$$= \frac{59q^2 - 154q + 107}{288(1-q)^3} + \frac{7q+9}{32(1+q)^2} + \frac{1-q}{8(1+q^2)} + \frac{q+2}{9(q^2+q+1)}.$$

Moreover the above partial decomposition can be expressed as

$$\frac{1/24}{(1-q)^3} + \frac{1/16}{(1-q)^2} + \frac{1/3}{1-q} + \frac{1/4}{(1-q^2)^2} + \frac{1/16 - q/4 - 3q^2/16}{1-q^4}$$

$$+ \frac{1}{4} \times \frac{(1+q^2)}{1-q^4} - \frac{1}{3} \times \frac{q+q^2}{1-q^3}.$$

Put

$$\sum_{n=0}^{\infty} \epsilon_n q^n = \frac{1}{4} \times \frac{(1+q^2)}{1-q^4} - \frac{1}{3} \times \frac{q+q^2}{1-q^3}.$$

Then we see $\epsilon_n \in \left\{-\frac{1}{3}, -\frac{1}{12}, 0, \frac{1}{4}\right\}$ and satisfy $-\frac{1}{2} < \epsilon_n < \frac{1}{2}$ for any n. Thus we have shown

$$\sum_{n=0}^{\infty} p(n \mid \text{parts in } \{2,3,4\}) q^n$$

$$= \frac{1}{24}\sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} q^n + \frac{1}{16}\sum_{n=0}^{\infty}(n+1)q^n + \frac{1}{3}\sum_{n=0}^{\infty} q^n$$

$$+\frac{1}{4}\sum_{n=0}^{\infty}(n+1)q^{2n} + \sum_{n=0}^{\infty} \frac{1/16 - q/4 - 3q^2/16}{1-q^4} + \sum_{n=0}^{\infty} \epsilon_n q^n.$$

On the other hand, one can easily verify that

$$\sum_{n=0}^{\infty} \left(\frac{(n+3)^2}{12} - \left[\frac{n+5}{4}\right]\left[\frac{n+3}{4}\right] \right)$$

$$= \frac{1}{24}\sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} q^n + \frac{1}{16}\sum_{n=0}^{\infty}(n+1)q^n + \frac{1}{3}\sum_{n=0}^{\infty} q^n$$

$$+\frac{1}{4}\sum_{n=0}^{\infty}(n+1)q^{2n} + \sum_{n=0}^{\infty} \frac{1/16 - q/4 - 3q^2/16}{1-q^4}.$$

Combining these formulas, we have

$$\sum_{n=0}^{\infty} p(n \mid \text{parts in } \{2,3,4\}) q^n = \sum_{n=0}^{\infty} \left(\frac{(n+3)^2}{12} - \left[\frac{n+5}{4}\right]\left[\frac{n+3}{4}\right] \right) + \sum_{n=0}^{\infty} \epsilon_n q^n.$$

Since $p(n \mid \text{parts in } \{2,3,4\})$ is an integer and $|\epsilon_n| < \frac{1}{2}$, we see

$$p(n \mid \text{parts in } \{2,3,4\}) = \left\{ \frac{(n+3)^2}{12} - \left[\frac{n+5}{4}\right]\left[\frac{n+3}{4}\right] \right\}$$

$$= \left\{ \frac{(n+3)^2}{12} \right\} - \left[\frac{n+5}{4}\right]\left[\frac{n+3}{4}\right],$$

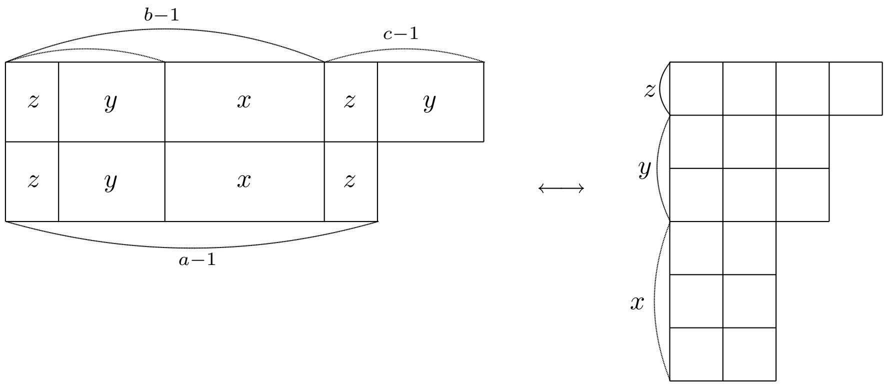
which completes a direct proof of the formula of $p(n \mid \text{parts in } \{2,3,4\})$, which is an answer to the exercise 84 in [3]. One knows $\triangle_n = p(n-3 \mid \text{parts in } \{2,3,4\})$ as follows. From the definition, we have

$$\triangle_n = p(n \mid\, n = a+b+c, a \geq b \geq c \geq 1, b+c > a).$$

Then we have the bijections:

$$
\begin{aligned}
&(a,b,c) &&\text{such that } \begin{cases} a+b+c=n \\ a\geq b\geq c\geq 1, b+c>a, \end{cases} \\
\longleftrightarrow\ &(a_1,b_1,c_1) &&\text{such that } \begin{cases} a_1+b_1+c_1=n-3 \\ a_1\geq b_1\geq c_1\geq 0, b_1+c_1\geq a_1, \end{cases} \\
\longleftrightarrow\ &(x,y,z) &&\text{such that } \begin{cases} 2x+3y+4z=n-3 \\ x\geq 0, y\geq 0, z\geq 0, \end{cases}
\end{aligned}
$$

where $a_1 = a-1, b_1 = b-1, c_1 = c-1$ and $x = b_1-c_1, y = b_1-c_1-a_1, z = a_1-b_1$. Here we shall show the relation of (a,b,c) and (x,y,z) by diagrams:



Thus we have verified the formula

$$\triangle_n = p(n-3 \mid \text{parts in } \{2,3,4\}) = \left\{\frac{n^2}{12}\right\} - \left[\frac{n}{4}\right]\left[\frac{n+2}{4}\right]$$

directly, which is an answer to the exercise 85 in [3].

References

[1] G. E. Andrews, A note on partitions and triangles with integer sides, *American Mathematical Monthly*, **86**, (1979), 477-478.

[2] G. E. Andrews, MacMahon's partition analysis: II, *Annals of Combinatorics*, **4**, (2000), 327-338.

[3] G. E. Andrews and K. Eriksson, Integer Partitions, Cambridge University Press, Cambridge 2004.