A Lower Decay Estimate for a Degenerate Kirchhoff Type Wave Equation with Strong Dissipation

By

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Abstract

Consider the initial-boundary value problem for the degenerate Kirchhoff type wave equation with strong dissipation : $\rho\dfrac{\partial^2 u}{\partial t^2} - \left(\displaystyle\int_\Omega |\nabla u(x,t)|^2\,dx\right)\Delta u - \delta\Delta\dfrac{\partial u}{\partial t} = 0$. For all $t \geq 0$, a lower decay estimate of the solution $\|\nabla u(t)\|^2 \geq c\,(1+t)^{-1}$ is derived when either the coefficient ρ or the initial data are appropriately smaller than the coefficient δ.

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1 Introduction

We consider the initial-boundary value problem for the following degenerate wave equation of Kirchhoff type with a strong dissipative term :

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\int_\Omega |\nabla u(x,t)|^2\,dx\right)\Delta u - \delta\Delta\frac{\partial u}{\partial t} = 0 \qquad \text{in} \quad \Omega \times [0,+\infty) \tag{1}$$

with the initial and boundary conditions

$$u(x,0) = u_0(x)\,, \quad \frac{\partial u}{\partial t}(x,0) = u_1(x) \qquad \text{in} \quad \Omega$$

and

$$u(x,t) = 0 \qquad \text{on} \quad \partial\Omega \times [0,+\infty)\,,$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $\Delta = \nabla \cdot \nabla = \sum_{j=1}^{N} \partial^2/\partial x_j^2$ is the Laplace operator, $\rho > 0$ and $\delta > 0$ are constants.

Matos and Pereira [1] have shown the existence of a unique global solution $u(t)$ in the class $L^\infty(0,T;H_0^1(\Omega)) \cap W^{1,\infty}(0,T;L^2(\Omega))$ with $u'(t) \in L^2(0,T;H_0^1(\Omega))$ for any $T > 0$, under the assumption that the initial data $\{u_0, u_1\}$ belong to $H_0^1(\Omega) \times L^2(\Omega)$. Moreover, by using the energy method, the energy decay estimate has been derived :

$$E(t) \equiv \rho\|u_t\|^2 + \frac{1}{2}\|\nabla u(t)\|^4 \leq C(1+t)^{-2}$$

for $t \geq 0$, where $u_t = \partial u/\partial t$ and $\|\cdot\|$ is the norm of $L^2(\Omega)$ (see [1, 3, 6]).

Concerning other upper decay estimates of the solution $u(t)$, in previous paper [6], we have already derived that

$$\|\nabla u_t(t)\|^2 \leq C(1+t)^{-3} \quad \text{and} \quad \|u_{tt}(t)\|^2 \leq C(1+t)^{-5}$$

for $t \geq 0$, under the assumption that the initial data $\{u_0, u_1\}$ belong to $\left(H^2(\Omega) \cap H_0^1(\Omega)\right) \times \left(H^2(\Omega) \cap H_0^1(\Omega)\right)$.

On the other hand, Nishihara [4] have derived a lower decay estimate of the solution $u(t)$: If the initial data $\{u_0, u_1\}$ belong to $\left(H^3(\Omega) \cap H_0^1(\Omega)\right) \times \left(H^3(\Omega) \cap H_0^1(\Omega)\right)$ and satisfy $\|\nabla u_0\|^2 + 2\rho(u_0, u_1) > 0$ and the initial energy $E(0) \equiv \rho\|u_1\|^2 + \dfrac{1}{2}\|\nabla u_0\|^4$ is sufficiently small, there exists a large time $T_* > 0$ such that

$$\|\nabla u(t)\|^2 \geq c\,(1+t)^{-1} \qquad \text{for} \quad t \geq T_* \tag{2}$$

with $c > 0$ (also see [2, 5, 6]).

Our purpose in this paper is to derive the lower decay estimate (2) for all $t \geq 0$ and to give a sufficient condition related to the size of the coefficient ρ and the initial data $\{u_0, u_1\}$ together with the coefficient δ.

We put

$$c_* \equiv \sup\left\{ \frac{\|v\|}{\|\nabla v\|} \;\middle|\; v \in H_0^1(\Omega)\,,\, v \neq 0 \right\}.$$

Our main result is as follows.

Theorem 1.1 *Let the initial data $\{u_0, u_1\}$ belong to $H_0^1(\Omega) \times H_0^1(\Omega)$ and $u_0 \neq 0$. Suppose that*

$$(3c_*)^2\rho\left(\rho\frac{\|\nabla u_1\|^2}{\|\nabla u_0\|^2} + \|\nabla u_0\|^2\right) < \delta^2\,. \tag{3}$$

Then, the solution $u(t)$ of (1) satisfies

$$c\,(1+t)^{-1} \leq \|\nabla u(t)\|^2 \leq C(1+t)^{-1} \qquad \textit{for} \quad t \geq 0 \tag{4}$$

where c and C are positive constants depending on the initial data $\{u_0, u_1\}$.

The proof of Theorem 1.1 is given by using Proposition 2.1 and Proposition 2.2 in the next section.

The notations we use in this paper are standard. The symbol $(\cdot, \cdot)$ means the inner product in $L^2(\Omega)$. Positive constants will be denoted by C and will change from line to line.

2 Lower decay

Proposition 2.1 *Let $u(t)$ be a solution of* (1) *and $M(t) \equiv \|\nabla u(t)\|^2 > 0$ for $0 \le t < T$. If $c_*(\rho H(0))^{1/2} < \delta$, then it holds that*

$$H(t) \le H(0) \qquad \text{for} \quad 0 \le t < T \tag{5}$$

where

$$H(t) \equiv \rho \frac{\|u_t(t)\|^2}{M(t)} + M(t)\,.$$

Proof. Multiplying (1) by $2u_t(t)$ and $M(t)^{-1}$, and integrating it over Ω, we have that

$$\begin{aligned}
\frac{d}{dt}H(t) + 2\delta\frac{\|\nabla u_t(t)\|^2}{M(t)} &= -\rho\frac{M'(t)}{M(t)^2}\|u_t(t)\|^2 \\
&\le 2c_*\rho\left(\frac{\|u_t(t)\|^2}{M(t)}\right)^{1/2}\frac{\|\nabla u_t(t)\|^2}{M(t)} \\
&\le 2c_*\left(\rho H(t)\right)^{1/2}\frac{\|\nabla u_t(t)\|^2}{M(t)}
\end{aligned}$$

and from the Young inequality that

$$\frac{d}{dt}H(t) + 2\left(\delta - c_*(\rho H(t))^{1/2}\right)\frac{\|\nabla u_t(t)\|^2}{M(t)} \le 0$$

for $0 \le t < T$.

If $c_*(\rho H(0))^{1/2} < \delta$, then we obtain

$$c_*(\rho H(t))^{1/2} \le \delta$$

for some $t > 0$, and

$$\frac{d}{dt}H(t) \le 0 \quad \text{or} \quad H(t) \le H(0)$$

for some $t > 0$. Thus we arrive at the desired estimate (5) for $0 \le t < T$. □

Proposition 2.2 *Let $u(t)$ be a solution of* (1) *and $M(t) > 0$ for $0 \le t < T$. If $3c_*(\rho H(0))^{1/2} < \delta$, then*

$$M(t) \equiv \|\nabla u(t)\|^2 \ge c\,(1+t)^{-1} \tag{6}$$

for $0 \le t < T$, where c is a positive constant depending on $\{u_0, u_1\} \in H_0^1(\Omega) \times H_0^1(\Omega)$.

Proof. Multiplying (1) by $2u_t(t)$ and $M(t)^{-3}$, and integrating it over Ω, we have that

$$\begin{aligned}
&\frac{d}{dt}\left(\rho\frac{\|u_t(t)\|^2}{M(t)^3} + \frac{1}{M(t)}\right) + 2\delta\frac{\|\nabla u_t(t)\|^2}{M(t)^3} \\
&\quad = -3\rho\frac{M'(t)}{M(t)^4}\|u_t(t)\|^2 - 2\frac{M'(t)}{M(t)^2} \\
&\quad \le 6c_*\rho\left(\frac{\|u_t(t)\|^2}{M(t)}\right)^{1/2}\frac{\|\nabla u_t(t)\|^2}{M(t)^3} + 4\left(\frac{\|\nabla u_t(t)\|^2}{M(t)^3}\right)^{1/2} \\
&\quad \le 6c_*\left(\rho H(t)\right)^{1/2}\frac{\|\nabla u_t(t)\|^2}{M(t)^3} + 4\left(\frac{\|\nabla u_t(t)\|^2}{M(t)^3}\right)^{1/2}
\end{aligned}$$

and from (5) that

$$\begin{aligned}
&\frac{d}{dt}\left(\rho\frac{\|u_t(t)\|^2}{M(t)^3} + \frac{1}{M(t)}\right) + 2\left(\delta - 3c_*(\rho H(0))^{1/2}\right)\frac{\|\nabla u_t(t)\|^2}{M(t)^3} \\
&\quad \le 4\left(\frac{\|\nabla u_t(t)\|^2}{M(t)^3}\right)^{1/2}
\end{aligned} \tag{7}$$

for $0 \le t < T$.

If $3c_*(\rho H(0))^{1/2} < \delta$, then we observe from (7) together with the Young inequality that

$$\frac{d}{dt}\left(\rho\frac{\|u_t(t)\|^2}{M(t)^3} + \frac{1}{M(t)}\right) \le C$$

and

$$\rho\frac{\|u_t(t)\|^2}{M(t)^3} + \frac{1}{M(t)} \le C(1+t)$$

for $0 \le t < T$ which gives the desired estimate (6). □

Proof of Theorem 1.1. Since $M(0) \equiv \|\nabla u_0\|^2 > 0$, putting

$$T \equiv \sup\left\{t \in [0, +\infty) \;\middle|\; M(s) > 0 \quad \text{for} \quad 0 \le s < t\right\},$$

we see that $T > 0$ and $M(t) > 0$ for $0 \leq t < T$. If $T < +\infty$, then it holds that $M(T) = 0$. However, from the lower estimate (6) we observe that $\lim_{t\to T} M(t) \geq c\,(1+T)^{-1} > 0$, and hence, we obtain that $T = +\infty$ and

$$M(t) > 0 \qquad \text{for all} \quad t \geq 0\,.$$

Thus, from (6) we have

$$M(t) \equiv \|\nabla u(t)\|^2 \geq c\,(1+t)^{-1}$$

for $t \geq 0$. On the other hand, by the standard energy method, we have

$$E(t) \equiv \|u_t(t)\|^2 + \frac{1}{2}\|\nabla u(t)\|^4 \leq C(1+t)^{-2}$$

for $t \geq 0$ where C is a positive constant depending on $\{u_0, u_1\} \in H_0^1(\Omega)\times L^2(\Omega)$. □

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