On a Classical-Quantum Correspondence for Mechanics in a Gauge Field

By

Ruishi Kuwabara

Department of Mathematical Sciences,
The University of Tokushima, Tokushima 770-8502, JAPAN

e-mail: kuwabara@ias.tokushima-u.ac.jp

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Abstract

This paper studies the classical and the quantum mechanics in a non-abelian gauge field on the basis of the symplectic geometry and the theory of representation of Lie groups. As a classical-quantum correspondence we present a conjecture on the quasi-mode corresponding to a certain classical energy level.

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Introduction

Let \((M, m)\) be a \(d\) dimensional smooth Riemannian manifold without boundary, and let \(\pi : P \rightarrow M\) be a principal \(G\)-bundle, where \(G\) is a compact semisimple Lie group with \(\dim G = r\). Suppose \(P\) is endowed with a connection \(\nabla\). The connection \(\nabla\) is defined by a \(g\)-valued one form (called the connection form) \(\theta\) on \(P\) with certain properties, where \(g\) is the Lie algebra of \(G\). The \(g\)-valued two form \(\Theta := d\theta + \theta \wedge \theta\) on \(P\) is called the curvature form of \(\nabla\). (See [4], for example.)

Take an open covering \(\{U_\alpha\}\) of \(M\) with \(\{\varphi_{\alpha \beta}\}\) being the transition functions of \(P\). Then the curvature form \(\Theta\) is regarded as a family of \(g\)-valued two forms \(\Theta_\alpha\) defined on \(U_\alpha\) such that

\[
\Theta_\beta = \text{Ad}(\varphi_{\alpha \beta}^{-1})\Theta_\alpha
\]

(0.1)
on \(U_\alpha \cap U_\beta(\neq \phi)\), where \(\text{Ad}(\cdot)\) denotes the adjoint action of \(G\) on \(g\). Such a family of \(g\)-valued two forms \(\{\Theta_\alpha\}\) on \(M\) satisfying (0.1) is called a gauge field, while the connection form \(\theta\) is called a gauge potential. If \(G\) is the abelian group \(U(1)\), then \(\Theta_\alpha = \Theta_\beta\) holds, and accordingly we have a two form \(\Theta\) globally defined on \(M\), which is called a magnetic field.

In this paper we study the classical and the quantum mechanics in the non-abelian gauge field \(\{\Theta_\alpha\}\) on the basis of the symplectic geometry and the
theory of representation of Lie groups. Section 1 is devoted to reviewing a geometrical formulation for the classical mechanics in the gauge field, which is essentially the same as that in the previous paper [6] (see also [7]). In Section 2 we introduce the space of quantum states corresponding to the classical system with an integral “charge”. (Related arguments are found in [8], [9].) Finally in Section 3 we present a conjecture on the quasi-mode corresponding to a certain classical energy level. This conjecture is a generalization of the eigenvalue theorem given in [5] for the abelian gauge field (the magnetic field).

1 Classical mechanics in a gauge field

1.1 The Kaluza-Klein metric

Let \((\cdot, \cdot)\) denote the inner product given by \((-1) \times \) (the Killing form) on the compact semisimple Lie algebra \(g(= T_v G)\), and let \(m_G\) be the metric on the Lie group \(G\) induced from \((\cdot, \cdot)_g\). Note that \(m_G\) is invariant under left- and right-translations on \(G\).

The connection \(\nabla\) on the principal bundle \(\pi: P \to M\) defines the direct decomposition of each tangent space \(T_p P (p \in P)\) as

\[
T_p P = H_p \oplus V_p,
\]

where \(V_p\) is tangent to the fiber, and \(H_p\) is linearly isomorphic with \(T_{\pi(p)} M\) through \(\pi_*|_{H_p}\). Note that the tangent space \(V_p\) to the fiber is linearly isomorphic with \(\mathfrak{g}\) by the correspondence \(g \ni A \mapsto P^p := \frac{d}{dt}(p \cdot \exp t A)|_{t=0} \in V_p\). The inner product on \(\mathfrak{g}\) induces the inner product \((\cdot, \cdot)_V\) on \(V_p (p \in P)\) as \((A^p, B^p)_V = (A, B)_g\). On the other hand, we have the inner product \((\cdot, \cdot)_{H_p}\) on \(H_p\) from the metric \(m\) on \(M\) such that \(\pi_*|_{H_p}\) is an isometry. Finally, we define an inner product \(\hat{m}\) in each \(T_p P (p \in P)\) by defining \(H_p\) and \(V_p\) to be orthogonal to each other. The metric \(\hat{m}\) on \(P\) (which is induced from the metric \(m\) on \(M\), the metric \(m_G\) on \(G\), and the connection \(\nabla\)) is called the Kaluza-Klein metric (cf. [3]). Note that \(\hat{m}\) is invariant under the \(G\)-action on \(P\).

Let \(\Omega_P = d\omega_P\) be the standard symplectic structure on the cotangent bundle \(T^* P\) of \(P\), where \(\omega_P\) is called the canonical one form on \(T^* P\). We have the natural Hamiltonian function \(\hat{H}\) on \(T^* P\) defined by the Kaluza-Klein metric \(\hat{m}\), i.e., \(\hat{H}(q) = \|q\|^2 (q \in T^* P)\). Thus, we have the Hamiltonian system \((T^* P, \Omega_P, \hat{H})\), which is just the system of geodesic flow on \(T^* P\).

1.2 Reduction of the system (cf. [1, Ch.4])

The action \(p \mapsto p \cdot g = R_g(p) (p \in P, g \in G)\) of \(G\) on \(P\) is naturally lifted to the action \(R^*_g := (R_{g^{-1}})^*\) on \(T^* P\) (so that \(R^*_g : T^*_p P \to T^*_{p \cdot g} P\) for each \(p \in P\)), which preserves \(\omega_P\) (and accordingly \(\Omega_P\)), i.e., \(R^*_g \cdot \omega_P = \omega_P\) holds for every \(g \in G\). (We call such action a symplectic action.) Moreover, we notice that the Hamiltonian \(\hat{H}\) is also invariant under the action \(R^*_g\).
A momentum map for the symplectic $G$-action $R^*_{g-1}$ is a map $J : T^*P \to g^*$ (the dual space of $g$) given by
\[ J(q,A) = (q_p, A_p^\nu) \quad (q \in T^*P, q_p \in T^*_p P (p \in P)), \]
for all $A \in g$. The momentum map $J$ is $\text{Ad}^*$-equivariant, i.e.,
\[ J \circ R^*_{g-1} = \text{Ad}^*(g^{-1}) \circ J \]
holds for $g \in G$, where $\text{Ad}^*(g) := (\text{Ad}(g^{-1}))^*$ (the adjoint of $\text{Ad}(g^{-1})$). Furthermore, $J$ is invariant under the flow of $(T^*P, \Omega_P, \tilde{H})$.

Note that $J$ is a surjective map with any $\mu \in g^*$ to be a regular value, and $J^{-1}(\mu)$ is a submanifold of $T^*P$. Put $G_\mu := \{g \in G; \text{Ad}^*(g) = \mu\}$, which is a closed subgroup of $G$. Then, $J^{-1}(\mu)$ is $G_\mu$-invariant because of (1.3). The quotient manifold $P_\mu := J^{-1}(\mu)/G_\mu$ is naturally endowed with a symplectic structure $\Omega_\mu$ induced from $\Omega_P$, and endowed with a Hamiltonian function $H_\mu$ induced from $\tilde{H}$. Thus we have a (reduced) Hamiltonian system $\mathcal{H}_\mu = (P_\mu, \Omega_\mu, H_\mu)$, which we regard as the dynamical system of classical particle of “charge” $\mu$ in the gauge field given by the connection $\tilde{\nabla}$ (the gauge potential). We remark that the reduced phase space $P_\mu$ is also given as the quotient manifold $J^{-1}(O_\mu)/G$ for the coadjoint orbit $O_\mu = \{\text{Ad}^*(g)\mu; g \in G\}$ in $g^*$.

1.3 A formulation by using the connection form

Suppose $G_\mu \subsetneq G$. Consider the quotient manifold $M_\mu := P/G_\mu$, and the natural projection $\pi' : M_\mu \to M(= P/G)$ gives a bundle structure with the fiber $G/G_\mu(\cong O_\mu)$. Let $\pi'^*_M : M_\mu^\# \to M_\mu$ be the vector bundle obtained by pulling back the cotangent bundle $T^*M$ over $M$ through the map $\pi' : M_\mu \to M$, i.e.,
\[ M_\mu^\# = \{(y, \xi) \in M_\mu \times T^*M; \pi'(y) = \pi_M(\xi)\}. \]
We note that $M_\mu^\#$ is regarded as a subbundle of $T^*M_\mu$ by the immersion $(y, \xi) \mapsto \pi'^*(\xi) \in T^*_y M_\mu$.

Let $\theta$ be the connection form (which is a $g$-valued one form on $P$) of $\tilde{\nabla}$, and put $\theta_\mu = \langle \mu, \theta \rangle$, which is an $\mathbb{R}$-valued one form on $P$.

**Lemma 1** Let $\mathfrak{g}_\mu$ be the Lie algebra of $G_\mu$. An element $A$ in $\mathfrak{g}$ belongs to $\mathfrak{g}_\mu$ if and only if $d\theta_\mu(A^P, X) = 0$ for any vector field $X$ on $P$.

**Proof.** We have
\[ d\theta_\mu(A^P, X) = (i(A^P)d\theta_\mu)(X) = (\mathcal{L}_{A^P}\theta_\mu)(X) - d(i(A^P)\theta_\mu)(X), \]
where $i(A^P)$ and $\mathcal{L}_{A^P}$ denote the interior product and the Lie derivative, respectively. Since $i(A^P)\theta_\mu = \theta_\mu(A^P) = \langle \mu, A \rangle = \text{constant}$, we have $d\theta_\mu(A^P, X) = 0$ for any vector field $X$ on $P$.
\[(\mathcal{L}_{AP}\theta_{\mu})(X)\]. Note that \(R_{g}^{\ast}\theta = \text{Ad}(g^{-1})\theta\) for \(g \in G\), and we get
\[(\mathcal{L}_{AP}\theta_{\mu})(X) = \frac{d}{dt}\langle\mu, \text{Ad}(\exp(-tA))(\theta(X))\rangle|_{t=0} = \frac{d}{dt}\langle\text{Ad}^{\ast}(\exp tA)\mu, \theta(X)\rangle|_{t=0}.
\]
This formula implies the assertion of the lemma.

By virtue of this lemma \(d\theta_{\mu}\) is regarded as a closed two form on \(M_{\mu}\). We introduce a two form
\[
\Omega_{\#_{\mu}} := (\tilde{\pi}')^{\ast}\Omega_{M} + (\pi'_{M_{\mu}})^{\ast}(d\theta_{\mu})
\]
on \(M_{\mu}^{\#}\), where \(\tilde{\pi}' : M_{\mu}^{\#} \to T^{\ast}M\) is the natural lift of \(\pi' : M_{\mu} \to M\), and \(\Omega_{M}\) is the standard symplectic form on \(T^{\ast}M\). The two form \(\Omega_{\#_{\mu}}\) is closed and non-degenerate, and accordingly defines a symplectic structure on \(M_{\mu}^{\#}\).

**Remark** The symplectic structure \(\Omega_{\#_{\mu}}\) is just the restriction of the twisted symplectic form \(\Omega_{M_{\mu}} + (\pi'_{M_{\mu}})^{\ast}(d\theta_{\mu})\) on \(T^{\ast}M_{\mu}\) to the subbundle \(M_{\mu}^{\#}\), where \(\pi'_{M_{\mu}} : T^{\ast}M_{\mu} \to M_{\mu}\) is the natural projection.

Let \(H\) be the Hamiltonian function on \(T^{\ast}M\) defined by the Riemannian metric \(m\) on \(M\), and put \(H_{\#_{\mu}} := (\tilde{\pi}')^{\ast}H + \|\mu\|^{2}\), where the norm \(\|\mu\|\) is naturally defined by the inner product \(m_{g}\) on \(g\). Thus we obtain the Hamiltonian system \((M_{\mu}^{\#}, \Omega_{\#_{\mu}}, H_{\#_{\mu}})\) (see Figure 1).

**Proposition 2** The Hamiltonian system \(\mathcal{H}_{\mu}\) is isomorphic with \((M_{\mu}^{\#}, \Omega_{\#_{\mu}}, H_{\#_{\mu}})\), that is, there exists a diffeomorphism \(\chi_{\mu} : P_{\mu} \to M_{\mu}^{\#}\) such that
\[
\Omega_{\mu} = \chi_{\mu}^{\ast}\Omega_{\#_{\mu}}, \quad H_{\mu} = \chi_{\mu}^{\ast}H_{\#_{\mu}}.
\]
Proof. For each $p \in P$ we put

\begin{align*}
(V^\perp)_p & := \{ q \in T^*_p P \mid \langle q, A_p^\mu \rangle = 0 \text{ for } \forall A \in \mathfrak{g} \} \subset T^*_p P), \\
(V^\perp_\mu)_p & := \{ q \in T^*_p P \mid \langle q, A_p^\mu \rangle = 0 \text{ for } \forall A \in \mathfrak{g}_\mu \} \subset T^*_p P,
\end{align*}

and define the subbundles $V^\perp := \bigcup_{p \in P}(V^\perp)_p$ and $V^\perp_\mu := \bigcup_{p \in P}(V^\perp_\mu)_p$ of $T^*P$, which are invariant under the $G_\mu$-action. Moreover we see that

$$M^\#_\mu \cong V^\perp/G_\mu, \quad T^*M_\mu \cong V^\perp_\mu/G_\mu.$$ 

For each $q \in T^*_p P$ we define the map

$$\bar{\chi}_\mu(q) := q - (\theta_\mu)_p \in T^*_p P.$$

Then, we see that

(i) $\bar{\chi}_\mu(q) \in (V^\perp)_p$ if $q \in J^{-1}(1_\mu)$, and that

(ii) $\bar{\chi}_\mu(R_{g^{-1}}^\#(q)) = R_{g^{-1}}^\#(\bar{\chi}_\mu(q))$ for $q \in J^{-1}(1_\mu)$ and $g \in G_\mu$.

Indeed, (i) is shown as follows: $(q_\mu, A_p^\mu) - ((\theta_\mu)_p, A_p^\mu) = (J(q), A) - (\mu, A) = 0$ for $\forall A \in \mathfrak{g}$. The assertion (ii) follows from the formula $(\theta_\mu)_p \cdot g = R_{g^{-1}}^\#((\theta_\mu)_p)$ $(g \in G_\mu)$, that is derived from the property $R_{g^{-1}}^\# \theta = \text{Ad}(g) \theta$ $(g \in G)$ for $\theta$ and the definition of $G_\mu$. Noticing (i) and (ii), we can define the diffeomorphism $\chi_\mu : P_\mu \to M^\#_\mu$ from map $\bar{\chi}_\mu : T^*P \to T^*P$.

Now, we will prove (1.4 a). A vector $X \in T_q(T^*P)$ $(q \in T^*P, \pi_P(q) = p)$ is written as

$$X(q) = \bar{X}(q) + X^*(q) \quad \text{with} \quad \bar{X}(q) \in T^*_p P, \quad X^*(q) \in T^*_p P(= T_q(T^*_p P)).$$

Then, $X^*(q) \in (V^\perp)_p$ if $X \in T_q J^{-1}(1_\mu)$. Let us take two vector fields $X = X(q)$ and $Y = Y(q)$ on $J^{-1}(1_\mu)$ defined in a neighborhood of $q_0 \in J^{-1}(1_\mu)$ such that $\bar{X}(q)$ and $\bar{Y}(q)$ are constant along the each fibers of $T^*P$. Then we have

$$\Omega_P(X, Y) = \frac{1}{2} \{ X(\omega_P, Y) - Y(\omega_P, X) - \langle \omega_P, [X, Y] \rangle \}$$

$$= \frac{1}{2} \{ X(q, \bar{Y}) - Y(q, \bar{X}) - \langle q, [\bar{X}, \bar{Y}] \rangle \}.$$

Put $q' = \bar{\chi}_\mu(q) = q - \theta_\mu(\in (V^\perp)_p)$, and we have

$$\Omega_P(X, Y) = \frac{1}{2} \{ X(q', \bar{Y}) - Y(q', \bar{X}) - \langle q', [\bar{X}, \bar{Y}] \rangle \}$$

$$+ \frac{1}{2} \{ \bar{X}(\theta_\mu, \bar{Y}) - \bar{Y}(\theta_\mu, \bar{X}) - \langle \theta_\mu, [\bar{X}, \bar{Y}] \rangle \}.$$

Here we notice that $\bar{X}(p') = \bar{X}(p)$ and $[\bar{X}, \bar{Y}] = [\bar{X}, \bar{Y}]$ hold. Therefore we see that the first term of this formula is regarded as $\Omega_M((\bar{\pi}' \circ \chi_\mu)_*([X]), (\bar{\pi}' \circ \chi_\mu)_*([Y]))$, and the second is regarded as $d\theta_\mu((\pi'_M \circ \chi_\mu)_*([X]), (\pi'_M \circ \chi_\mu)_*([Y]))$. 

Finally we prove (1.4 b). Take $q \in T^*_P \cap J^{-1}(\mu)$. Then, we have $q = \bar{x}_\mu(q) + (\theta_\mu)_a$ with $\bar{x}_\mu(q) \in (V^*)_p$, $(\theta_\mu)_p \in (H^*)_p$. Since $(V^*)_p$ and $(H^*)_p$ are orthogonal each other, we have

$$H_\mu([q]) = \|\bar{x}_\mu(q)\|^2 + \|((\theta_\mu)_p)\|^2 = H(\pi' \circ \chi_\mu([q])) + \|((\theta_\mu)_p)\|^2.$$ 

Here, $(\theta_\mu)_p(A^\mu_p) = (\mu, A)$ for $\forall A \in g$, and accordingly $\|((\theta_\mu)_p)\| = \|\mu\|$ holds. ■

Wong’s equation on $M_\mu$. We represent the flow of the system $(M^\#, \Omega^\#, H^\#_\mu)$ using local coordinates. Let $(x, g) = (x^1, \ldots, x^d, g^1, \ldots, g')$ be local coordinates of $U \times G \cong \pi^{-1}(U) \subset P$ for $U \subset M$. Note that $M_\mu$ is locally diffeomorphic with $U \times (G/G_\mu)$. Suppose the connection form $\theta$ of $\tilde{\nabla}$ is represented as

$$\theta(x, g) = \sum_{j=1}^d \theta_j(x, g)dx^j + \sum_{\alpha=1}^r \theta_\alpha(x, g)dg^\alpha.$$ 

Then, the curvature form $\Theta := d\theta + \theta \wedge \theta$ of $\tilde{\nabla}$ is locally written as

$$\Theta(x, g) = \frac{1}{2} \sum_{i,j} \theta_{ij}(x, g)dx^i \wedge dx^j = \frac{1}{2} \sum_{i,j} \left\{ \left( \frac{\partial \theta_j}{\partial x^i} - \frac{\partial \theta_i}{\partial x^j} + [\theta_i, \theta_j] \right)dx^i \wedge dx^j. \right.$$ 

Put $\Theta_\mu := (\mu, \Theta)$, and it is shown similarly to $d\theta_\mu$ that $\Theta_\mu$ is an $\mathbb{R}$-valued two form globally defined on $M_\mu$. We get the following by straightforward calculations.

**Proposition 3** The motion of the particle in the system $(M^\#, \Omega^\#, H^\#_\mu)$ is governed by the equation (called Wong’s equation [7]) on $M_\mu$ locally expressed as

$$\ddot{x}^i + \sum_{j,k} \Gamma^i_{jk}(x)\dot{x}^j \dot{x}^k - 2 \sum_{j,k} m_{ij}(x)\Theta^{(\mu)}_{jk}(x, g)\dot{x}^j \dot{x}^k = 0$$

$$\ddot{g} + L_{g^*}\left( \sum_{j} \theta_j(x, g)\dot{x}^j \right) = 0$$

where $\Theta^{(\mu)}_{jk}(x, g) := \langle \mu, \Theta_{jk}(x, g) \rangle$, $\Gamma^i_{jk}(x)$ denotes Christoffel’s symbol on the Riemannian manifold $(M, m)$, and $L_{g^*} : g(=T_eG) \to T_eG$ is the left translation. (Note that $\Theta^{(\mu)}_{jk}(x, g)$ and the second equation is invariant under $G_\mu$-action, namely they depend only on the equivalent class $[g] \in G/G_\mu$.)

2 Quantum systems in a gauge field

2.1 Unitary representations of $G$ and the quantum states
Let \( \mathfrak{g}_C \) be the complexification of the Lie algebra \( \mathfrak{g} \). Let \( \mathfrak{h} \) denote a Cartan subalgebra of \( \mathfrak{g}_C \), and let \( R \) be the root system for the pair \( (\mathfrak{g}_C, \mathfrak{h}) \). Put \( \mathfrak{h}_R := \{ H \in \mathfrak{h}; \ a(H) \in \mathbb{R} \text{ for } \forall a \in R \} \). Then, \( \mathfrak{h}_R = i \mathfrak{t}_C = \mathfrak{h} \) holds for a Cartan subalgebra \( \mathfrak{t} \) of \( \mathfrak{g} \). We notice that \( \mathfrak{h}_R \) is a \( \ell(= \text{rank} \, G) \) dimensional real vector space with the inner product \( (iH, iH')_K = -(H, H')_K = (H, H')_{\mathfrak{g}} \) \( (H, H' \in \mathfrak{t}) \), where \( (\cdot, \cdot)_K \) denotes the Killing form on \( \mathfrak{g}_C \) (or \( \mathfrak{g} \)). By identifying \( \mathfrak{g} \) to \( \mathfrak{g}^* \) with respect to the inner product \( (\cdot, \cdot)_\mathfrak{g} \) we have \( \mathfrak{h}_R^* = i\mathfrak{t}^* \subset i\mathfrak{g}^* \). Put \( \Gamma := t^* \text{exp}^{-1}(e) \) for \( \exp : \mathfrak{g}_C \to G_C \), where \( G_C \) is the simply connected Lie group whose Lie algebra is \( \mathfrak{g}_C \). Then, \( \Gamma \) is a lattice in \( \mathfrak{t} \cong \mathbb{R}^\ell \). Let \( \Gamma^* \) be the dual lattice of \( \Gamma \), namely

\[
\Gamma^* = \{ \tau \in \mathfrak{t}^* \mid (\tau, H) \in 2\pi \mathbb{Z} \text{ for } \forall H \in \Gamma \}.
\]

Then, \( i\Gamma^* \) is a lattice in \( i\mathfrak{t}^* = \mathfrak{h}_R^* \), whose element is called an integral form. Let \( C \) be a Weyl chamber in \( \mathfrak{h}_R^* \). Then \( C \) defines the set \( \Gamma^+ \) of positive roots and the ordering in \( \mathfrak{h}_R^* \). The set \( \hat{G} \) of irreducible unitary representations is labeled by the set \( \Gamma \cap i\Gamma^+ \) (whose element is called a dominant integral form).

For a “charge” \( \mu \in \mathfrak{g}^* \) the coadjoint orbit \( O_\mu \) in \( \mathfrak{g}^* \) intersects the set \( i\mathfrak{t} \) in exactly one point \( i\lambda \) \( (\lambda \in C) \). We assume that \( \lambda \) lies on \( i\Gamma^* \setminus \{0\} \), i.e., \( \lambda \) is integral. We call such \( \mu \) a quantized charge. Let \( (\rho_\lambda, V_\lambda) \) be the irreducible unitary representation of \( G \) with highest weight \( \lambda \). We introduce the associated vector bundle \( \mathcal{E}_\lambda = P \times_{\rho_\lambda} V_\lambda \to M \) of \( P \) through the representation \( (\rho_\lambda, V_\lambda) \). We regard the Hilbert space \( L^2(M, \mathcal{E}_\lambda) \) of \( L^2 \)-sections of \( \mathcal{E}_\lambda \) as the space of quantum states corresponding to the classical system \( \mathcal{H}_\mu \) for the quantized charge \( \mu \).

The connection \( \nabla \) on \( P \) induces the covariant derivative \( \nabla^{(\lambda)} : C^\infty(M, \mathcal{E}_\lambda) \to C^\infty(M, T^*M \otimes \mathcal{E}_\lambda) \) on \( \mathcal{E}_\lambda \), and we obtain the Laplacian \( \Delta^{(\lambda)} := (\nabla^{(\lambda)})^* \nabla^{(\lambda)} : L^2(M, \mathcal{E}_\lambda) \to L^2(M, \mathcal{E}_\lambda) \), which is a non-negative, (formally) self-adjoint, second order elliptic differential operator.

Let \( s : U(\subset M) \to P \) be a local section of \( P \), and set \( \theta_U := s^* \theta \) for the connection form \( \theta \) of \( \nabla \). Suppose \( \theta_U \) is expressed as \( \sum A_j(x)dx^j \) \( (A_j(x) \in \mathfrak{g}) \). Then, the covariant derivative \( \nabla^{(\lambda)}(\theta_U) \) is given by

\[
\nabla^{(\lambda)}_j f = \nabla_j f + A_j^{(\lambda)}(x)f \quad (f \in C^\infty(U, V_\lambda))
\]

with \( A_j^{(\lambda)}(x) = (\rho_\lambda)_*(A_j(x)) \in \mathfrak{u}(V_\lambda) \), and

\[
\Delta^{(\lambda)} = -\sum_{j,k} m^{jk}(x)(\nabla_j + A_j^{(\lambda)}(x))(\nabla_k + A_k^{(\lambda)}(x))
\]

where \( \nabla \) is the Levi-Civita connection on \( (M, m) \).

### 2.2 Spaces of \( L^2 \) Functions on \( P \) and \( L^2 \) Sections of \( \mathcal{E}_\lambda \)

Let \( L^2_\lambda(P, V_\lambda) \) be the space of \( V_\lambda \)-valued \( L^2 \) functions \( f \)'s on \( P \) satisfying

\[
f(p \cdot g) = \rho_\lambda(g^{-1})f(p) \quad (p \in P)
\]
for any $g \in G$. Then, we have the natural unitary isomorphism (by taking suitable inner products):

$$L^2(M,E_\lambda) \cong L^2_\lambda(P,V_\lambda).$$

Let $\chi_\lambda$ denote the character of the representation $\rho_\lambda$, and define the map $P_\lambda : L^2(P) \to L^2(P)$; $f \mapsto f_\lambda$ by

$$f_\lambda(p) := d_\lambda \int_G \chi_\lambda(g^{-1}) \overline{f(p \cdot g)} \, dg \quad (p \in P),$$

where $d_\lambda := \dim V_\lambda$, and $dg$ is the Haar measure on $G$. Let $L^2_\lambda(P)$ be the image of $P_\lambda$. Using local coordinates, $P \ni \pi^{-1}(U) \ni p = (x,g) \in U \times G$, we can see that $L^2_\lambda(P)$ consists of functions locally expressed as

$$f_\lambda(p) = f_\lambda(x,g) = \sum_{j,k} [\rho_\lambda(g)]^j_k f_0(x)^j_k$$

for some functions $f_0(x)^j_k$ on $U$, where $[\rho_\lambda(g)]^j_k$ denotes the matrix-components of the representation $\rho_\lambda$. By virtue of the Peter-Weyl theorem we have

$$L^2(P) = \bigoplus_{\rho_\lambda \in \hat{G}} L^2_\lambda(P).$$

Define the map $F_\lambda : L^2(P) \to L^2(P,V_\lambda^* \otimes V_\lambda)$; $f \mapsto F_\lambda$ by

$$F_\lambda(p) := d_\lambda \int_G f(p \cdot g) \rho_\lambda(g) \, dg \quad (p \in P).$$

Here $\rho_\lambda(g)$ is regarded as a element of $V_\lambda^* \otimes V_\lambda = \text{End}_\mathbb{C}(V_\lambda)$, and we have a local expression

$$F_\lambda(p) = F_\lambda(x,g) = \rho_\lambda(g^{-1}) F_0(x)$$

for a matrix-valued function $F_0(x)$ on $U$. We denote by $L^2_\lambda(P,V_\lambda^* \otimes V_\lambda)$ the image of the map $F_\lambda$. Then, we have the following.

**Lemma 4** The function $F \in L^2(P,V_\lambda^* \otimes V_\lambda)$ belongs to $L^2_\lambda(P,V_\lambda^* \otimes V_\lambda)$ if and only if

$$F(p \cdot g) = \rho_\lambda(g^{-1}) F(p) \quad (p \in P)$$

(2.2) holds for any $g \in G$.

Proof. The “only if”-part of the statement is shown by directly checking (2.2). Suppose $F$ satisfies (2.2). Then, $F$ is locally expressed as $F(x,g) = \rho_\lambda(g^{-1})K(x)$ for some matrix-valued function $K(x)$. Take the $L^2$ function $f$ on $P$ (locally) defined by

$$f(x,g) = \text{Trace} [\rho_\lambda(g^{-1}) \, K(x)].$$
Then, we have $\mathcal{F}_\lambda(f) = F$.

Let $\{v_j\}_{j=1}^{d_\lambda}$ be a orthonormal basis of $V_\lambda$. It follow from the above lemma that the $V_\lambda$-valued functions $f_j^\lambda(p) := F_\lambda(p)v_j$ $(j = 1, \ldots, d_\lambda)$ belong to $L^2_\lambda(P, V_\lambda)$. As a result we have the following isomorphism:

$$L^2_\lambda(P, V_\lambda^* \otimes V_\lambda) \cong \prod_{j=1}^{d_\lambda} L^2_\lambda(P, V_\lambda).$$

Finally, for $F_\lambda \in L^2_\lambda(P, V_\lambda^* \otimes V_\lambda)$ (which is a matrix-valued function) we define $[\Phi_\lambda(F_\lambda)](p) := \text{Trace}^t\left[ \Phi_\lambda(F_\lambda)(p) \right] (p \in P)$. Then, $\mathcal{P}_\lambda = \Phi_\lambda \circ \mathcal{F}_\lambda$ holds, and $\Phi_\lambda$ is a bijection from $L^2_\lambda(P, V_\lambda^* \otimes V_\lambda)$ onto $L^2_\lambda(P)$. In fact, for $f(x, g) = \sum_{j,k}[\rho_\lambda(g)]_k f_j^\lambda(x) \in L^2_\lambda(P)$ (locally), we have $[\Phi_\lambda^{-1}f](x, g) = \rho_\lambda(g^{-1}) F(x)$ for the $(d_\lambda \times d_\lambda)$ matrix $F(x) := \{f(x)_j^\lambda\}$.

As a consequence, we get the following one-to-one correspondences:

$$L^2_\lambda(P) \cong L^2_\lambda(P, V_\lambda^* \otimes V_\lambda) \cong L^2_\lambda(P, V_\lambda) \oplus \cdots \oplus L^2_\lambda(P, V_\lambda) \cong L^2(M, \mathcal{E}_\lambda) \oplus \cdots \oplus L^2(M, \mathcal{E}_\lambda),$$

that is, more explicitly

$$\sum \oplus L^2(M, \mathcal{E}_\lambda) \quad \sum \oplus L^2_\lambda(P, V_\lambda) \quad L^2_\lambda(P, V_\lambda^* \otimes V_\lambda) \quad L^2_\lambda(P)$$

$(\psi_1, \ldots, \psi_{d_\lambda}) \mapsto (\psi_1, \ldots, \psi_{d_\lambda}) \mapsto \Psi = (\psi_1, \ldots, \psi_{d_\lambda}) \mapsto \psi_P = \text{Trace}^t\left[ \Psi \right].$

Let $\Delta_P$ be the Laplace-Beltrami operator on $(P, \tilde{m})$. Then, $\Delta_P$ leaves $L^2_\lambda(P)$ invariant. Notice that the Laplace-Beltrami operator $\Delta_G$ on $(G, m_G)$ satisfies

$$\Delta_G[\rho_\lambda(g)]_k^\lambda = (|\lambda + \delta|^2_K - |\delta|^2_K)\rho_\lambda(g)]_k^\lambda,$$

where $\delta = \frac{1}{2} \sum_{\alpha \in \mathbb{R}^+} \alpha \in \mathfrak{h}_K^\perp$ and the norm $||K$ (and the inner product $(\cdot, \cdot)_K$) on $\mathfrak{h}_K^\perp$ is naturally induced one from that on $\mathfrak{h}_K$, and we have the following lemma by the formula (2.1).

**Lemma 5** Suppose $L^2_\lambda(P) \ni \psi_P \mapsto \psi_j \in L^2(M, \mathcal{E}_\lambda)(j = 1, \ldots, d_\lambda)$ is the above correspondence. Then, we have

$$(\Delta_P \psi_P)_j = \Delta_\lambda \psi_j + (|\lambda + \delta|^2_K - |\delta|^2_K)\psi_j.$$

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We assume that $M$ is compact. Then, the spectrum of $\Delta^{(\lambda)}$ consists of non-negative eigenvalues

$$\nu_1^{(\lambda)} \leq \nu_2^{(\lambda)} \leq \cdots \leq \nu_k^{(\lambda)} \leq \cdots \uparrow +\infty.$$ 

If $\psi \in L^2_\lambda(P)$ satisfies $\Delta P\psi = \kappa \psi$, then

$$\Delta^{(\lambda)}\psi_j = (\kappa - \|\lambda + \delta\|_K^2 + \|\delta\|_K^2)\psi \quad (j = 1, \ldots, d_\lambda).$$

Conversely, suppose $\psi \in L^2(M, \mathcal{E}_\lambda)$ satisfies $\Delta^{(\lambda)}\psi = \nu\psi$. Put

$$\Psi^{(j)} = (0, \ldots, 0, \psi, 0, \ldots, 0) \quad (j = 1, \ldots, d_\lambda).$$

Then, $\psi_P^{(j)} = \text{Trace}[\Psi^{(j)}] \in L^2_\lambda(P)$ satisfies

$$\Delta_P\psi_P^{(j)} = (\nu + \|\lambda + \delta\|_K^2 - \|\delta\|_K^2)\psi_P^{(j)}.$$ 

Thus, we have the following for the spectrum $\{\nu_j^{(\lambda)}\}$ of $\Delta^{(\lambda)}$ and that of $\Delta_P$.

**Proposition 6** The spectrum of $\Delta_P$ is the set of eigenvalues given by

$$\bigcup_{\lambda \in \mathcal{G}} d_\lambda \cdot \{ \nu_j^{(\lambda)} + \|\lambda + \delta\|_K^2 - \|\delta\|_K^2 \mid j \in \mathbb{N} \},$$

where $d_\lambda \cdot \{ \}$ denotes the set of $d_\lambda$ copies of $\{ \}$, and

$$d_\lambda = \prod_{\alpha \in \mathcal{H}_+} \frac{(\lambda + \alpha, \alpha)_K}{(\delta, \alpha)_K}.$$

### 3 Quasi-mode for the mechanics in a gauge field

#### 3.1 Quantum energies associated to a Lagrangian manifold

Suppose $\mu \in \mathfrak{g}^*$ is a quantized charge, namely, $i\lambda = \mathcal{O}_\mu \cap i\mathfrak{c}$ belongs to $i\mathfrak{g}^* \setminus \{0\}$. We have a quantum system associated to $\mathcal{H}_\mu = (M^\#_\mu, \Omega^\#_\mu, H^\#_\mu)$, that is a quantum Hamiltonian given by

$$\hat{H}_\lambda = \Delta^{(\lambda)} + \|\lambda + \delta\|_K^2$$

$$= - \sum_{j,k} m^j(x)(\nabla_j + A_j^{(\lambda)}(x))(\nabla_k + A_k^{(\lambda)}(x)) + \|\lambda + \delta\|_K^2.$$

acting on $L^2(M, \mathcal{E}_\lambda)$. For the element $\lambda \in \mathcal{C} \cap \Gamma^*$ let us consider the “ladder” of representations with the highest weights $\{n\lambda; n \in \mathbb{N}\}$ and the associated family of quantum systems $(\hat{H}_n\lambda, L^2(M, \mathcal{E}_n\lambda))$. 

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In the case of abelian gauge group $U(1)$ we established in [5] a eigenvalue theorem for the magnetic Schrödinger operator, which asserts the existence of an approximate quantum energy associated to a certain classical energy level. We here present the following conjecture which is a generalization of the eigenvalue theorem to the case of non-abelian gauge group $G$.

**Conjecture** Suppose there exists a compact Lagrangian submanifold $L_P$ of $(T^*P, \Omega_P)$ contained in $J^{-1}(O_\mu)$. Let $L = \chi_\mu \circ \pi_{O_\mu}(L_P)$, which is a submanifold of $M^\#$. Assume the following conditions:

(i) $H^\#_\mu \equiv e$ on $L$ for a real constant $e$,

(ii) $L_P$ is invariant under the Hamiltonian flow $\varphi_t$ on $(T^*P, \Omega_P, \bar{H})$, and the restricted flow $\varphi|_{L_P}$ leaves invariant a non-zero half-density on $L_P$, and

(iii) (quantization condition) for every closed curve $\gamma$ on $L_P$,

$$\frac{1}{2\pi} \int_\gamma \omega_P - \frac{1}{4} m_{L_P}([\gamma]) \in \mathbb{Z}$$

(3.1)

holds, where $m_{L_P} \in H^1(L_P, \mathbb{Z})$ is the Maslov class of $L_P$.

Let $d$ be the smallest element of the set $\{1, 2, 4\}$ for which $d \cdot m_{L_P}([\gamma]) \equiv 0 \pmod{4}$ for all $[\gamma] \in \pi_1(L_P)$, and set

$$n_k := dk + 1, \quad \tilde{n}_k := \frac{1}{2} \left( n_k + \frac{\|n_k \lambda + \delta\|_K}{\|\lambda\|_K} \right)$$

for $k \in \mathbb{N} \cup \{0\}$. (Note that $\tilde{n}_k \sim n_k$ as $k \to \infty$.)

Then, there is a sequence $\{E_{j_k}^{(n_k \lambda)}\}_{k=0}^\infty$ of eigenvalues of $\hat{H}_{n_k \lambda}$ such that

$$E_{j_k}^{(n_k \lambda)} = e \tilde{n}_k^2 + O(1) \quad (k \to \infty).$$

(3.2)

**Observation** Put $h = 1/\tilde{n}_k$, and consider the Schrödinger operator

$$\hat{H}(h) := \frac{1}{\tilde{n}_k^2} \hat{H}_{n_k \lambda}$$

depending on the Planck constant $h$. Then, $E(h) := E_{j_k}^{(n_k \lambda)}/\tilde{n}_k^2$ is an eigenvalue of $\hat{H}(h)$, and the formula (3.2) means that

$$E(h) = e + O(h^2)$$

as $h \to 0$. Thus, we see that the classical energy $e$ obtained by the quantization condition gives an approximation of a quantum energy of order $h^2$ in a semi-classical sense.

### 3.2 Plan to prove the conjecture
Let 
\[ \tilde{G} := S^1 \times G = \{ (e^{it}, g); 0 \leq t < 2\pi, g \in G \}. \]

The strategy to prove the conjecture is to construct a suitable operator \( A : \mathcal{D}'(\tilde{G}) \to \mathcal{D}'(P) \) (where \( \mathcal{D}'(\cdot) \) denotes the space of distributions). The idea is essentially due to [11] by Weinstein, and applied in [5] in the case of magnetic flow, i.e., \( G = U(1) \).

By virtue of the Peter-Weyl each element \( u(t, g) \) in \( L^2(\tilde{G}) \) is written as
\[ u(t, g) = \sum_{\ell \in \mathbb{Z}} \sum_{\rho \in \hat{G}} \sum_{j,k} \hat{u}_{\ell \rho}^{jk} e^{i\ell t} [\rho(g)]_{jk}^l. \] (3.3)

For the sequence \( \{ n_k \}_{k=0}^\infty \) \( (n_k = dk + 1) \) we define the subspace \( L^2(\tilde{G}; \{ n_k \lambda \}) \) of \( L^2(\tilde{G}) \) as follows: A function \( u \in L^2(\tilde{G}) \) written as (3.3) belongs to \( L^2(\tilde{G}; \{ n_k \lambda \}) \) if and only if \( \hat{u}_{\ell \rho}^{jk} = 0 \) holds for every \( \ell, \rho \notin \{ (n_k, n_k \lambda) \}_{k=0}^\infty \).

Put \( D_G := (\Delta_G + \| \delta \|_K)^{1/2} \), which is a first order pseudodifferential operator satisfying
\[ D_G[\rho n \lambda(g)]_{jk}^l = (\| n \lambda + \delta \|_K) [\rho n \lambda(g)]_{jk}^l \] (\( n \in \mathbb{N} \)).

Let us consider a continuous linear operator \( A : \mathcal{D}'(\tilde{G}) \to \mathcal{D}'(P) \) which satisfies the following conditions:

(A-i) \( e^{-1} \Delta_P A - AD_G \) induces a bounded operator from \( L^2(\tilde{G}) \) to \( L^2(P) \), where
\[ D_G := -\frac{1}{4} \left( \frac{\partial}{\partial t} + \frac{i}{\| \lambda \|_K} D_G \right)^2. \]

(A-ii) \( A : L^2(\tilde{G}; \{ n_k \lambda \}) \to L^2(P) \) is an isometry.

(A-iii) Take
\[ (u_k^l(t, g)) := \sqrt{\frac{d_k}{2\pi}} e^{in_k \lambda} [\rho n_k \lambda(g)]_{jk}^l \] \( (d_k := \dim V_{n_k \lambda}) \)
in \( L^2(\tilde{G}; \{ n_k \lambda \}) \). Then, \( \psi_k(\psi_k)^l := A[u_k^l] \) belongs to \( L^2(n_k \lambda(P), V_{n_k \lambda} \otimes V_{n_k \lambda}) \).

Suppose we have the above operator \( A \). Note that
\[ D_G u_k = \tilde{n}_k^2 u_k. \]

By virtue of (A-i) we have
\[ \|(e^{-1} \Delta_P - \tilde{n}_k^2) \psi_k \|_{L^2(P)} = \|(e^{-1} \Delta_P A - AD_G) u_k \|_{L^2(P)} \leq M \| u_k \|_{L^2(\tilde{G})} = M, \] (3.4)
$M$ being a constant. Let $\{\varphi_j^{(k)}\}$ be the orthonormal basis of eigenfunction of $\Delta_P|_{L^2_{nk}(P)}$. By means of Lemma 5 we have

$$\Delta_P \varphi_j^{(k)} = \tilde{E}_j^{(nk\lambda)} \varphi_j^{(k)}$$

with

$$\tilde{E}_j^{(nk\lambda)} = E_j^{(nk\lambda)} - \|\delta\|^2_K.$$  \hfill (3.5)

Using the expansion: $\psi_k = \sum_j \hat{\psi}_j \varphi_j^{(k)}$, we have

$$\| (e^{-1} \Delta_P - \tilde{n}_k^2) \psi_k \|_{L^2(P)}^2 = \| e^{-1} \sum_j \hat{\psi}_j \tilde{E}_j^{(nk\lambda)} \varphi_j^{(k)} - \sum_j \tilde{n}_k^2 \hat{\psi}_j \varphi_j^{(k)} \|_{L^2(P)}^2$$

$$= \frac{1}{e^2} \sum_j \{ \tilde{E}_j^{(nk\lambda)} - e \tilde{n}_k^2 \}^2 |\hat{\psi}_j|^2$$

$$\geq \frac{1}{e^2} \min_j \{ \tilde{E}_j^{(nk\lambda)} - e \tilde{n}_k^2 \}^2 \sum_j |\hat{\psi}_j|^2$$

$$= \frac{1}{e^2} \min_j \{ \tilde{E}_j^{(nk\lambda)} - e \tilde{n}_k^2 \}^2.$$  

Note $\sum_j |\hat{\psi}_j|^2 = 1$ by means of (A-ii). Combining this inequality with (3.4), we have

$$\min_j \{ \tilde{E}_j^{(nk\lambda)} - e \tilde{n}_k^2 \}^2 \leq e^2 M,$$

that is

$$| \tilde{E}_j^{(nk\lambda)} - e \tilde{n}_k^2 | = \min_j | \tilde{E}_j^{(nk\lambda)} - e \tilde{n}_k^2 | \leq \text{Const.}$$ \hfill (3.6)

We obtain the formula (3.2) from (3.5) and (3.6). The sequence $\{(\psi_k, e \tilde{n}_k^2)\}_{k=0}^\infty$ in this argument is called a quasi-mode of $\Delta_P$ (cf. [2]).

Thus, a proof of the conjecture is carried out if we can construct the operator $A$ and check the properties (A-i)-(A-iii). We expect that this procedure will be similarly performed as [5] (see also [10], [11]) by constructing the operator $A$ as a Fourier integral operator under the quantization condition (3.1).

References


