Nimstring Values for $2 \times n$ Rectangular Arrays I

By

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Abstract

Games of Nimstring are deeply related to games of Dots and Boxes. To understand Dots and Boxes, it is important to analyze games of Nimstring. But only a few study on them has been done. We get the Nimstring values for some $2 \times n$ rectangular arrays in this paper.

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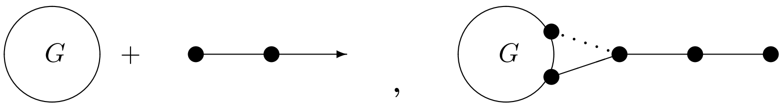
Introduction

A dual form of *Dots-and-Boxes* is *Strings-and-Coins*, with strings, coins and scissors. The ends of each piece of string are glued to two different coins or to a coin and the ground. Each string has at most one end glued to the ground, and each player in turn cuts a new string. If your cut completely detaches a coin, you pocket it and must then cut another string if there are still uncut strings. The game ends when all coins are detached, and the player who pockets the greater number is the winner. The coins and strings form the nodes and edges of a graph. The game of *Nimstring* is played on the same kind of graphs as Strings-and-Coins, and you make exactly the same move by cutting a string, which is a complimenting move whenever you detach a coin. Nimstring is played according to the *normal play rule*, that is, you lose when you detach the last coin, for then rules require you to make a further move when it is impossible to do so. Nimstring games are *impartial*. Hence their values must be *nimbers*. But, we need to notice the following. Let G be any

Nimstring game with no capturable coins. Then its value $|G|$ ie determined by the *mex(minimal excluded)rule*, that is, it is the least number 0 or 1 or 2 or $\cdots$ that is not among the numbers of the values of the graphs left after cutting single strings. The value of a graph with a caputurable coin of one of the next two types is the *loony value* "L".

Figure 1



When adding the loony value, we have

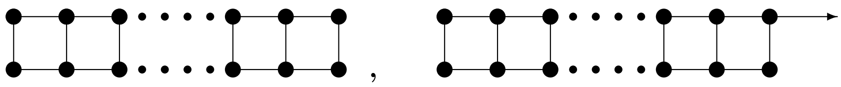
$$"L" + *m = "L" + "L" = "L".$$

The value of a graph with capturable coins of other types is equal to that of the subgraph obtained by removing the capturable coins and strings.

1 Rectangular arrays

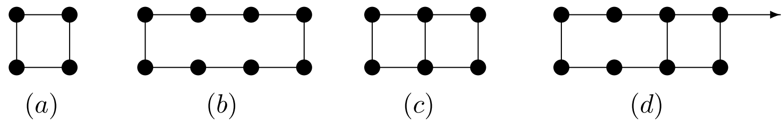
In the present paper, the Nimstring values of $2 \times n$ rectangular arrays of the following two types are mainly studied. We use little arrows that run to the ground.

Figure 2



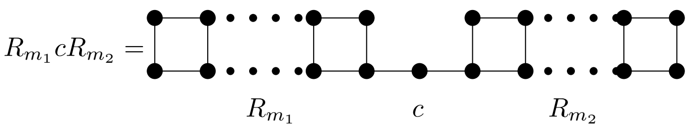
Next, we give some simple examples of rectangular arrays.

Figure 3



In this paper, A rectangular like (a) or (b) in Figure 3 is called a *box*. We define the *size* of the box (a) as 1 and that of (b) as 3. The numbers of boxes of rectangular arrays $(a), (b), (c)$ and (d) are 1,1, 3 and 2 respectively. The Nimstring values of a $2 \times n$ rectangular arrays mainly depends on the number of boxes contained in it. An ungrounded $2 \times (m+1)$ rectangular array with m boxes is described as R_m and an grounded one with an arrow a is described as $R_m a$. Hence, (a) and (b) are R_1 , (c) is R_3 and (d) is $R_2 a$. The main purpose of the present paper is to obtain the Nimstring values of R_m and $R_m a$ for any positive integer m. In general, we obtain the Nimstring value of a graph by removing its edge and using the *mex rule*. If we remove an edge of e of a graph and get a subgraph, we describe the value of the subgraph as $|e|$ simply. We should not remove a horizontal edge(simply, h-edge) of a box whose size is greater than 3, because if we do so, we get a *loony position*. By the same reason, we should not remove any edges at the ends. A vertical edge is written as a v-edge simply. A edge which is not at the end is said to be *inner*. Take a a rectangular R_m. Let D be one of its boxes with size k($1 \leq k \leq 3$). Assume D is not at the end. Remove a proper h-edge of D, we obtain the following.

Figure 4



Let the left part of the above graph be R_{m_1} and the right part of that be R_{m_2}. The part c connecting R_{m_1} and R_{m_2} is said to be its *connection of R_{m_1} and R_{m_2}* . We define the size of c to be the number of edges in c. We represent above graph as $R_{m_1} c R_{m_2}$.

Proposition 1. *The Nimstring value of R_m is 0 (resp. $*$) if m is odd (resp. even).*

Proof. By removing an inner h-edge, we get a subgraph $H = R_{n_1} c R_{n_2}$, where $n_1 + n_2 = m - 1$.

Assume m is odd. If we remove any inner v-edge, we get R_{m-1} whose value is $*$ by induction. The value $|c|$ in H, actually, the value of a proper edge of c is 0, for if n_1 and n_2 are odd (resp. even), we have $0+0=0$ (resp. $*+*=0$), by induction. Hence the value $|H|$ is not 0. Now R_m has some edges with value $*$ and the values of its all edges are not 0. Hence the value of R_m is 0.

Next, assume m is even. If we remove an inner v-edge, we get R_{m-1} whose value is 0. As the value $|c|$ in H is $*$, the value $|H|$ is not $*$. Hence, R_m has some edges with value 0 and the values of its all edges are not $*$. Thus, the value of R_m is $*$.

Proposition 2. *Let G be a graph $R_{m_1} c R_{m_2}$. Let the rightmost box of R_{m_1} be A with size a and the leftmost box of R_{m_2} be B with size b. Let the size of c be $k (1 \leq k \leq 3)$.*

(1) In the following cases, the value $|G|$ is 0 (resp. $*$) if $m_1 + m_2$ is odd (resp. even).

$(m_1 = m_2 = 1), (m_1 = b = k = 1), (m_1 = 1, b \geq 4), (m_2 = a = k = 1), (m_2 = 1, a \geq 4), (a = b = k = 1), (a \geq 4, b = k = 1), (a = 1, b \geq 4, k = 1), (a \geq 4, b \geq 4)$.

(2) In the other case, the value $|G|$ is $*2$ (resp. $*3$), if $m_1 + m_2$ is odd (resp. even).

Proof. By removing an inner h-edge of R_{m_1}, we get a subgraph $G_1 = R_{n_1} d R_{n_2} c R_{m_2}$, where $n_1 + n_2 = m_1$ and d is a connection of R_{n_1} and R_{n_2}. Assume we remove a proper inner h-edge and get a subgraph $G_2 = R_{m_1} c R_{\ell_1} e R_{\ell_2}$, where $\ell_1 + \ell_2 = m_2 - 1$ and e is a connection of R_{ℓ_1} and R_{ℓ_2}.

(1) Assume that $m_1 + m_2$ is odd. We may assume m_1 is odd and m_2 is even. The value $|c|$ is $*$. In the other cases than the case $(a \geq 4, b \geq 4)$, we can remove some edges of A and B, and the values of the edges are $*$ or $*3$. The values of the inner v-edges are also $*$ or $*3$. The value $|d|$ in G_1 is 0, for if n_1 and n_2 are odd (resp. even), the value of $R_{n_2} c R_{m_2}$ and that of R_{n_1} are 0 (resp. $*$). Hence the value $|G_1|$ is not 0. The value $|e|$ in G_2 is 0, for if ℓ_1 is odd (resp. even) and ℓ_2 is even (resp. odd), the both values of $R_{m_1} c R_{\ell_1}$ ant R_{ℓ_2} are $*$ (resp. 0). Hence, the value $|G_2|$ is not 0. In G, the values of all edges are not 0. Thus, when $m_1 + m_2$ is odd, the value $|G|$ is 0.

Assume both m_1 and m_2 have the same parity. Then the value $|c|$ is 0. The values of inner v-edges are 0 or $*2$. In the other cases than the case $(a \geq 4, b \geq 4)$, we can remove some edges of A and B and their values are 0 or $*2$. In this case, we show the value $|d|$ is $*$. Firstly, let m_1 and m_2 be odd. The value $|R_{n_2} c R_{m_2}|$ is $*$ (resp. 0) and $|R_{n_1}|$ is 0 (resp. $*$), if n_1 and n_2 are odd (resp.if n_1 and n_2 are even). Next, let m_1 and m_2 be even. The value $|R_{n_2} c R_{m_2}|$ is $*$ (resp. 0) and $|R_{n_1}|$ is 0 (resp. $*$), if n_1 is odd and n_2 is even (resp. if n_1 is even and n_2 is odd). Hence, the value $|G_1|$ is not $*$. Thus, since G has some edges with value 0 and the values of its all edges are not $*$, the value $|G|$ is $*$.

(2) Assume m_1 is odd and m_2 is even. The value $|c|$ is $*$. Suppose we remove a proper edge of A (resp. B) and get a subgraph $G_3 = R_{m_1-1}c'R_{m_2}$(resp. $G_4 = R_{m_1}c"R_{m_2-1}$). If the size of c' (resp. $c"$) is less than 4, the value $|G_3|$ (resp. $|G_4|$) is $*$ or $*3$. But, we always have G_3 (resp. G_4) in which the size of c' (resp. $c"$) is greater than 3. In this case, the values of $G_3|$ and G_4 are 0. The value of any inner edges of R_{m_1} or R_{m_2} are $*3$. We show the value $|d|$ in G_1 is $*2$. If both n_1 and n_2 are odd (resp. even), the value $|R_{n_2}cR_{m_2}|$ is $*2$ (resp. *3) and $|R_{m_1}|$ is 0 (resp. $*$). The value $|G_1|$ is not $*2$. We show the value $|e|$ in G_2 is $*2$. If ℓ_1 is odd (resp. even) and ℓ_2 is even (resp. odd), the value $|R_{m_1}cR_{\ell_1}|$ is $*3$ (resp. $*2$) and $|R_{\ell_2}|$ is $*$ (resp. 0). Hence, the value $|G_2|$ is not $*2$. In G, there are edges with value 0 and edges with value $*$, and the values of all its edges are not $*2$. Thus, the value $|G|$ is $*2$.

Assume m_1 and m_2 have the same parity. Then, the value $|c|$ is 0. There exist a proper edge of A or B whose value is $*$. The values of edges of A and B are 0, $*$ or 2. The values of inner v-edges of R_{m_1} or R_{m_2} are $*2$. We show the value of $|d|$ in G_1 is $*3$ in this case. Firstly, Let m_1 and m_2 be odd. If n_1 and n_2 are odd (resp. even), the value $|R_{n_2}cR_{m_2}|$ is $*3$ (resp. *2) $|R_{m_1}|$ is 0 (resp. $*$). Next, let m_1 and m_2 be even. If n_1 is odd (resp. even) and n_2 is even (resp. odd), the value $|R_{n_2}cR_{m_2}|$ is $*3$ (resp. *2) and $|R_{m_1}|$ is 0 (resp. $*$). Hence, the value $|G_1|$ is not $*3$ in this case. In G, there are edges with value 0, ones with value $*$ and ones with value $*2$, and the values of all its edges are not $*3$. Thus, the value $|G|$ is $*3$.

2 Arrays with one arrow

We study the value of an array with one arrow $R_m a$. It has m boxes and one arrow denoted by a. Denote the rightmost box by A. Let the size of a be the number x of edges and a unit arrow. Let the size of A be y and that of the box next to A(if $m \geq 2$) be z.

Proposition 3. *Put $G = R_m a$.*

(1) If $x = 1, y = 1$ or $x = 1, 2, 3, y \geq 4$, the value $|R_m a|$ is $*$ (resp. 0) when m is odd (resp. even).

(2) If $x = 1, 2, 3$ and $y = 2, 3$ or $x = 2, 3$ and $y = 1$, the value $|R_m a|$ is $*3$ (resp. $*2$), when m is odd (resp. even).

Proof. By removing a proper inner h-edge, we get a subgraph $H = R_{n_1}cR_{n_2}a$ where $n_1 + n_2 = m - 1$ and c is a connection of R_{n_1} and $R_{n_2}a$.

(1) Let m be odd. The value $|a|$(correctly, the value of a proper edge of a) is 0, by Proposition 1. If $x = 1$ and $y = 1$, the value of a edge of A is

$*2$ by induction. The value of the rightmost inner v-edge is $*2$ (resp. 0) if $x = y = 1, z = 1, 2$ (resp. $x = y = 1, z \geq 3$ or $x = 1, 2, 3, y \geq 4$). The value of any other inner v-edge is 0. We show the value $|d|$ in H is $*$. If n_1 and n_2 are odd (resp. even), the value $|R_{n_1}|$ is 0 (resp. $*$) and $|R_{n_2}a|$ is $*$ (resp. 0). Hence, the value $|H|$ is not $*$. Thus G has edges with value 0 and the values of all its edges are not $*$. Now we get the value $|G|$ is $*$.

Let m be even. Then, the value $|a|$ is $*$. If $x = 1$ and $y = 1$, the value of a edge of A is $*3$ by induction. The value of the rightmost inner v-edge is $*3$ (resp. $*$) if $x = y = 1, z = 1, 2$ (resp. $x = y = 1, z \geq 3$ or $x = 1, 2, 3, y \geq 4$). The values of any other edges are $*$. We show the value $|c|$ in H is 0 in this case. If n_1 is odd (resp. even) and n_2 is even (resp. odd), the values $|R_{n_1}|$ and $|R_{n-2}a|$ are 0 (resp. $*$). Hence the value $|H|$ is not 0. The value $|G|$ is 0, because the values of all its edges are not 0,

(2) Let m be odd. Then, the value $|a|$ is 0 and that of a upper h-edge of A is $*$. The value of a lower h-edge is $*2$ if $z = 1, 2, 3$ and $x = 1, y = 2$ or $x = 2, y = 1$, 0 if $z \geq 4$ and $x = 1, y = 2$ or $x = 2, y = 1$, and $*$ in the other cases. The value of the rightmost inner v-edge is $*2$ if $x = 1, 2, 3, y = 2, z = 1$ or $x = 2, 3, y = 1, z = 1, 2$, and 0 in the other cases. The values of the other inner v-edges are always $*2$. We show the value $|c|$ in H is $*3$ in this case. The value $|R_{n_1}|$ is 0 (resp. $*$) and $|R_{n_2}a|$ is $*3$ (resp. $*2$), if n_1 and n_2 are odd (resp. even). Hence, the value $|H|$ is not $*3$. In G, there are some edges with value 0, ones with value $*$ and ones with $*2$. Thus, the value $|G|$ is $*3$, because the values of all its edges are not $*3$.

Let m be even. The value $|a|$ is $*$ and that of a upper h-edge of A is 0. The value of a lower h-edge of A is $*3$ if $z = 1, 2, 3$ and $x = 1, y = 2$ or $x = 2, y = 1$, $*$ if $z \geq 4$ and $x = 1, y = 2$ or $x = 2, y = 1$ and 0 in other cases. The value of the rightmost inner v-edge is $*3$ if $x = 1, 2, y = 2$ and $z = 1$ or $x = 2, 3, y = 1$ and $z = 1, 2$, and $*$ in the other cases. We prove the value $|c|$ in H is $*2$. The value $|R_{n_1}|$ is 0 (resp. $*$) and $|R_{n_2}a|$ is $*2$ (resp. $*3$), if n_1 is odd (resp. even) and n_2 is even (resp. odd). Hence, the value $|H|$ is not $*2$. In G, there are some edges with value 0 and ones with value $*$. Thus, the value $|G|$ is $*2$, because the values of all its edges are not $*2$.

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