# Generalized Goggins's Formula for Lucas and Companion Lucas Sequences

By

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**Abstract**

In our previous papers, we have generalized Goggins's formula given in [1] into two different directions [2] and [3]. In this paper, we shall give a more generalized formula which combine the results in [2] and those in [3]. Our formula (6) involves our previous results (4), (5) and also Goggins's formula (1) as its special cases. Furthermore we shall give another formula (8) which is a generalization of a formula obtained in [2] too.

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# Introduction

In [1], J. G. Goggins has shown the following formula

$$\frac{\pi}{4} = \sum_{n=1}^{\infty} \tan^{-1}(1/F_{2n+1}), \tag{1}$$

where $F_n$ is the $n$th Fibonacci number. Since $F_1 = 1$ and $\pi/4 = \tan^{-1}(1/F_1)$, (1) is equivalent to the following formula

$$\frac{\pi}{2} = \sum_{n=0}^{\infty} \tan^{-1}(1/F_{2n+1}). \tag{2}$$

From the fact $F_{-2k-1} = F_{2k+1}$, (2) is also equivalent to the following formula

$$\pi = \sum_{n=-\infty}^{\infty} \tan^{-1}(1/F_{2n+1}). \tag{3}$$

In our previous paper [2], we have generalized this formula (3) to the following formula which holds for any integer $k$,

$$k\pi = \sum_{n=-\infty}^{\infty} \tan^{-1}(F_{2k}/F_{2n+1}). \tag{4}$$

In our previous paper [3], we gave the following formula which is another generalization of (2) for Lucas sequences

$$\frac{\pi}{2} = \sum_{n=0}^{\infty} \tan^{-1}(t/u_{2n+1}), \tag{5}$$

where $u_n$ is the Lucas sequences associated to the parameter $(t, -1)$. Namely $t$ is a positive integer with initial terms $u_0 = 0, u_1 = 1$ satisfying the binary recurrence sequence $u_n = tu_{n-1} + u_{n-2}$ for any $n \in \mathbf{Z}$.
In this paper, we shall combine the formulas (4) and (5). Actually we shall prove the following formula which holds for any integer $k$,

$$k\pi = \sum_{n=-\infty}^{\infty} \tan^{-1}(u_{2k}/u_{2n+1}). \tag{6}$$

In our paper [3], we have also proved the following formula,

$$\frac{\pi}{2} = \sum_{n=-\infty}^{\infty} \tan^{-1}(t/v_{2n}). \tag{7}$$

Let $k$ be any odd integer. In the last section, we shall generalize this formula (7) to the following formula

$$\frac{k\pi}{2} = \sum_{n=-\infty}^{\infty} \tan^{-1}(v_k/v_{2n}). \tag{8}$$

Here we note that one can verify (7) is the special case $t = v_1$, i.e., $k = 1$ of this formula (8).

## 1 Formulas for Lucas sequences

Let $t$ be a positive integer and $\{G_n\}$ be a binary recurrence sequence which satisfies

$$G_{n+2} = tG_{n+1} + G_n.$$

Using the induction on $m$, one can easily show the following addition theorem of $G_\ell$. Though one can see the proofs of this addition formula in [2] or [4], we will give a following simple proof for the sake of completeness of this paper.

**Addition Theorem**.

$G_{m+\ell} = u_m G_{\ell+1} + u_{m-1} G_\ell$, *for any integer m.*

Proof. Since $u_0 = 0$, $u_{-1} = u_1 = 1$, one can easily see that this formula is true for the cases $m = 0$ and $m = 1$. Assume the formula is true for the cases $m$ and $m - 1$. Then we have

$G_{m+1+\ell} = tG_{m+\ell} + G_{m-1+\ell}$
$= t(u_m G_{\ell+1} + u_{m-1} G_\ell) + (u_{m-1} G_{\ell+1} + u_{m-2} G_\ell)$
$= (tu_m + u_{m-1})G_{\ell+1} + (tu_{m-1} + u_{m-2})G_\ell. = u_{m+1} G_{\ell+1} + u_m G_\ell.$

Thus we have verified that the formula is true for the case $m + 1$.
Conversely, we know

$G_{m-2+\ell} = G_{m+\ell} - tG_{m-1+\ell}$
$= (u_m G_{\ell+1} + u_{m-1} G_\ell) - t(u_{m-1} G_{\ell+1} + u_{m-2} G_\ell)$
$= (u_m - tu_{m-1})G_{\ell+1} + (u_{m-1} - tu_{m-2})G_\ell. = u_{m-2} G_{\ell+1} + u_{m-3} G_\ell.$

Thus we have verified that the formula is also true for the case $m - 2$, which completes the proof of the addition theorem.

Substituting $G_{\ell+1} - G_{\ell-1}$ for $tG_\ell$, we have

$tG_{m+\ell} = tu_m G_{\ell+1} + u_{m-1}(G_{\ell+1} - G_{\ell-1}) = (tu_m + u_{m-1})G_{\ell+1} - u_{m-1} G_{\ell-1}$
$= u_{m+1} G_{\ell+1} - u_{m-1} G_{\ell-1}.$

Thus we have obtained a modified version of this addition theorem.

**Corollary 1**. $tG_{m+\ell} = u_{m+1} G_{\ell+1} - u_{m-1} G_{\ell-1}$, for any integer $m$.

Let us consider the special case when $G = u$ and $\ell$ is even and $m$ is odd in Corollary 1. Put $\ell = 2n$ and $m = 2k - 1$. Then we can write $tu_{2n+2k-1} = u_{2k} u_{2n+1} - u_{2k-2} u_{2n-1}$. Thus we have shown:

**Corollary 2**. $tu_{2n+2k-1} + u_{2k-2} u_{2n-1} = u_{2k} u_{2n+1}$.

Let us consider the special case when $G = u$, $\ell = 2n$ and $m = -2n - 2k + 2$ in

Corollary 1. Then we can show
$tu_{-2k+2} = u_{-2n-2k+3}u_{2n+1} - u_{-2n-2k+1}u_{2n-1}$,
which is equivalent to
$-tu_{2k-2} = u_{2n+2k-3}u_{2n+1} - u_{2n+2k-1}u_{2n-1}$.
Thus we have shown the following corollary.

**Corollary 3**. $u_{2n+2k-1}u_{2n-1} - tu_{2k-2} = u_{2n+2k-3}u_{2n+1}$.

Using these corollaries, we can show the following proposition.

**Proposition 1**.

$$\tan^{-1}\left(\frac{u_{2k-2}}{u_{2n+2k-1}}\right) + \tan^{-1}\left(\frac{t}{u_{2n-1}}\right) = \tan^{-1}\left(\frac{u_{2k}}{u_{2n+2k-3}}\right).$$

Proof. From Corollaries 2 and 3, we have

$$\frac{\dfrac{u_{2k-2}}{u_{2n+2k-1}} + \dfrac{t}{u_{2n-1}}}{1 - \dfrac{tu_{2k-2}}{u_{2n+2k-1}u_{2n-1}}} = \frac{u_{2k-2}u_{2n-1} + tu_{2n+2k-1}}{u_{2n+2k-1}u_{2n-1} - tu_{2k-2}} = \frac{u_{2k}u_{2n+1}}{u_{2n+2k-3}u_{2n+1}}$$

$$= \frac{u_{2k}}{u_{2n+2k-3}},$$

which completes the proof.

This proposition and the fact $\lim\limits_{n\to\pm\infty} \tan^{-1}(u_{2m}/u_{2n+1}) = 0$ for any fixed $m$ imply that

$$\sum_{n=-\infty}^{\infty} \tan^{-1}\left(\frac{u_{2k-2}}{u_{2n+1}}\right) + \sum_{n=-\infty}^{\infty} \tan^{-1}\left(\frac{t}{u_{2n-1}}\right) = \sum_{n=-\infty}^{\infty} \tan^{-1}\left(\frac{u_{2k}}{u_{2n+1}}\right).$$

Put $A(k) = \sum\limits_{n=-\infty}^{\infty} \tan^{-1}\left(\dfrac{u_{2k}}{u_{2n-1}}\right)$. Then the above relation can be rewritten as

$$A(k-1) + A(1) = A(k).$$

Here we note that $u_2 = t$ by definition and $A(1) = \pi$ from the formula (5). Therefore, using the induction on $k$, we can obtain the first formula (6) as follows.

**Theorem 1**. *With the above notations, we have*

$$\sum_{n=-\infty}^{\infty} \tan^{-1}(u_{2k}/u_{2n+1}) = k\pi,$$

*or equivalently*

$$\sum_{n=0}^{\infty} \tan^{-1}(u_{2k}/u_{2n+1}) = \frac{k\pi}{2}, \textit{ for any fixed integer } k.$$

**Remark 1**. From the facts $u_{-2n} = -u_{2n}$ and $tan^{-1}(-x) = -\tan^{-1}(x)$, we can see

$$\sum_{n=-\infty}^{\infty} \tan^{-1}(u_{2k}/u_{2n}) = 0, \text{ where } n \text{ runs all the integers except } 0.$$

Combining this fact and the above theorem, we have a modified version of the above formula

$$\sum_{n=-\infty}^{\infty} \tan^{-1}(u_{2k}/u_n) = k\pi, \text{ where } n \text{ runs all integers} \neq 0.$$

# 2 A formula for companion Lucas sequences

In the following, we shall restrict ourselves to the special case when $k$ is an odd positive integer at first. Put $\beta_{2n}(k) = \tan^{-1}(v_k/v_{2n})$ and $\beta_{2n-1} = \tan^{-1}(2/v_{2n-1})$ for any index $n$. Then we can show the following proposition.

**Proposition 2.** *For any integer $n \geq 1$,*
$2\beta_{2n}(k) = \beta_{2n-1} - \beta_{2n+1}$, *for the case* $2n \geq k+1$,
*and*
$2\beta_{2n}(k) = \pi + \beta_{2n-1} - \beta_{2n+1}$, *for the case* $2 \leq 2n \leq k-1$.

Proof. We have

$$\tan(\beta_{2n-k} - \beta_{2n+k}) = \frac{2/v_{2n-k} - 2/v_{2n+k}}{1 + 4/(v_{2n-k}v_{2n+k})} = \frac{2(v_{2n+k} - v_{2n-k})}{v_{2n+k}v_{2n-k} + 4}.$$

By virtue of Binet's formula, we have

$$v_{2n+k} - v_{2n-k} = (\varepsilon^{2n+k} + \bar{\varepsilon}^{2n+k}) - (\varepsilon^{2n-k} + \bar{\varepsilon}^{2n-k}) = (\varepsilon^{2n} + \bar{\varepsilon}^{2n})(\varepsilon^{k} + \bar{\varepsilon}^{k})$$
$$= v_k v_{2n},$$

where we used the elementary fact $\varepsilon^k \bar{\varepsilon}^k = (-1)^k = -1$.
We also have

$$v_{2n+k}v_{2n-k} + 4 = (\varepsilon^{2n+k} + \bar{\varepsilon}^{2n+k})(\varepsilon^{2n-k} + \bar{\varepsilon}^{2n-k}) + 4$$
$$= (\varepsilon^{4n} + \bar{\varepsilon}^{4n}) - (\varepsilon^{2k} + \bar{\varepsilon}^{2k}) + 4 = (\varepsilon^{4n} + \bar{\varepsilon}^{4n} + 2) - (\varepsilon^{2k} + \bar{\varepsilon}^{2k} - 2)$$
$$= (\varepsilon^{2n} + \bar{\varepsilon}^{2n})^2 - (\varepsilon^{k} + \bar{\varepsilon}^{k})^2 = v_{2n}^2 - v_k^2.$$

On the other hand, we have

$$\tan(2\beta_{2n}(k)) = \frac{v_k/v_{2n} + v_k/v_{2n}}{1 - (v_k/v_{2n})^2} = \frac{2v_k v_{2n}}{{v_{2n}}^2 - v_k^2}.$$

Thus we have shown $\tan(\beta_{2n-k} - \beta_{2n+k}) = \tan(2\beta_{2n}(k))$.

Hence we have $2\beta_{2n}(k) = \beta_{2n-k} - \beta_{2n+k} + m\pi$ for some integer $m$.
Since $0 < \beta_{2n}(k) < \pi/2$ and $|\beta_{2n-1}| < \pi/2$ for any $n$, we have more precisely

$$2\beta_{2n}(k) = \beta_{2n-k} - \beta_{2n+k}, \text{ for the case } 2n \geq k+1,$$

and

$$2\beta_{2n}(k) = \pi + \beta_{2n-k} - \beta_{2n+k}, \text{ for the case } 2 \leq 2n \leq k-1,$$

which completes the proof of the proposition.

Then, from the facts $v_{-2n} = v_{2n}$ and $v_{-2n-1} = -v_{2n+1}$, we have

$$\begin{aligned}
&\sum_{n=-\infty}^{\infty} \tan^{-1}(v_k/v_{2n}) = \tan^{-1}(v_k/v_0) + \sum_{n=1}^{\infty} 2\tan^{-1}(v_k/v_{2n}) \\
&= \tan^{-1}(v_k/2) + \sum_{n=1}^{\infty} 2\beta_{2n}(k) \\
&= \tan^{-1}(v_k/2) + (k-1)\pi/2 + \sum_{n=1}^{\infty}(\beta_{2n-k} - \beta_{2n+k}) \\
&= \tan^{-1}(v_k/2) + (k-1)\pi/2 \\
&\quad +(\beta_{-(k-2)} + \beta_{-(k-4)} + \cdots + \beta_{-1} + \beta_1 + \cdots + \beta_{k-4} + \beta_{k-2}) + \beta_k \\
&\quad +(\beta_{k+2} - \beta_{k+2}) + (\beta_{k+4} - \beta_{k+4}) + \cdots + (\beta_{k+2n} - \beta_{k+2n}) + \cdots \\
&= \tan^{-1}(v_k/2) + (k-1)\pi/2 + \beta_k \\
&= \tan^{-1}(v_k/2) + (k-1)\pi/2 + \tan^{-1}(2/v_k) = k\pi/2.
\end{aligned}$$

Thus we have shown the formula (8) for the case when $k$ is an odd positive integer.

Now we shall verify the case when $k$ is an odd negative integer. We note that $v_{-k} = -v_k$ for any odd integer k. Hence, for any odd negative integer $k$, we can also verify the formula (8) reduing the positive case $-k$ as follows.

$$\sum_{n=-\infty}^{\infty} \tan^{-1}(v_k/v_{2n}) = \sum_{n=-\infty}^{\infty} \tan^{-1}(-v_{-k}/v_{2n})$$

$$= -\left(\sum_{n=-\infty}^{\infty} \tan^{-1}(v_{-k}/v_{2n})\right) = -\left(\frac{-k\pi}{2}\right) = \frac{k\pi}{2}.$$

**Theorem 2**. *With the above notations, we have*

$$\sum_{n=-\infty}^{\infty} \tan^{-1}(v_k/v_{2n}) = \frac{k\pi}{2}, \textit{ for any odd integer } k.$$

**Remark 2**. From the fact $v_{-2n-1} = -v_{2n+1}(\neq 0)$, we have the following formula

$$\sum_{n=-\infty}^{\infty} \tan^{-1}(v_k/v_{2n+1}) = 0.$$

Combining the above theorem and this result, we can give another modified version of the formula (8) as follows

$$\sum_{n=-\infty}^{\infty} \tan^{-1}(v_k/v_n) = \frac{k\pi}{2}, \text{ for any odd integer } k. \tag{9}$$

## References


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