Study on the Phenomena of Potential Well in the View Point of Natural Statistical Physics

By

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Abstract

In this paper, we study on the phenomena of potential well in the view point of natural statistical physics.

We derive the Schrödinger equation of potential well by virtue of my new theory of natural statistical physics. This derivation is very new and natural and reasonable in the physical sense.

We solve the eigenvalue problem for the Schrödinger equation of potential well. Thereby we obtain the new and true eigenfunctions of the Schrödinger operator of the potential well.

By virtue of my new theory, we can understand the phenomena of potential well naturally and reasonably in the physical sense.

At last, we study the meaning of the impact force naturally and reasonably in the physical sense.

As for these results, we refer to Ito [1]∼[4]. Especially, as for the phenomena of potential well, we refer to Ito [2], [4].

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Introduction

In this paper, we study on the phenomena of potential well in the view point of natural statistical physics.

Using the mathematical model of a phenomenon of potential well, we can understand the motion of free electrons in a metallic conductor.

The object of this research is to establish the complete theory of natural statistical physics. For that purpose, we study the physical phenomena in this subject. It is important that the scale of the problems are large or small. But it is also important to construct the complete theory systematically. Though the themes of research are seen to be scattered, they take shape as the natural statistical physics. By studying this problem, it does not mean the closing of a series of research of natural statistical physics.

In order to understand the phenomenon, it is better to study the phenomenon truthfully.

We cannot understand the phenomenon only by thinking independently of the physical phenomenon as it is.

In order to understand the physical phenomenon, we do not think it logically, but we think it using the causality law.

Even if we consider all possible cases about the physical phenomenon logically, we have not the master card to determine which case is true effectively. In order to understand a physical phenomenon, it is important to study the phenomenon using the causality law.

Of course, when the theoretical model is determined correctly, we study the model mathematically using logical thinking.

1 Setting of the problem

Now we consider the case where an infinite number of electrons move in a small conductor. We can neglect the mutual interaction of electrons. The motion of electrons are controlled by some electro-magnetic force.

The case where the action of the force is expressed by a potential well approximately is a phenomenon of potential well.

Here we consider the good setting of the problem.

Now we consider that a physical system Ω is the family of electrons moving in a potential well $V(x)$ in one-dimensional space.

Here, we assume that an electron is a material point with mass m and electric charge $-e$. Therefore, the control by the action of electric force is expressed in the potential $V(x)$.

Assume that $V > 0, a > 0$ are two constant. Then the potential well $V(x)$ is expressed in the form

$$V(x) = \begin{cases} -V, & (-a \le x < a), \\ 0, & (x < a,\ x \ge a) \end{cases}$$

on 1-dimensional space $\boldsymbol{R}$.

Then, by virtue of the potential $V(x)$, the force

$$\boldsymbol{F} = -\frac{dV(x)}{dx} = V\delta_{-a}(x) - V\delta_a(x)$$

acts on the electrons.

Thereby, we controlled the motion of electrons. This means that the impact force V acts in the positive direction at the point $-a$, and the impact $-V$ acts in the negative direction at the point a.

Here,

$$\begin{aligned} V(x) &= -\int \boldsymbol{F}(x)dx = \int_{-\infty}^{x} \frac{dV(x)}{dx} dx \\ &= \int_{-\infty}^{x} \Big(-V\delta_{-a}(x) + V\delta_a(x) \Big) dx \\ &= \begin{cases} -V, & (-a \le x < a), \\ 0, & (x < -a,\ x \ge a) \end{cases} \end{aligned}$$

is the potential energy of the electron for the force $\boldsymbol{F}$.

Then we assume that L^2-density $\psi(x,t)$ determines the natural probability distribution of the position variable x of the physical system depending on the time t. Then the L^2-density $\psi(x,t)$ is the solution of the Schrödinger equation.

Here the graph of the probability density $|\psi(x,t)|^2$ determined by the L^2-density $\psi(x,t)$ is the graph which represents the mathematical information given by the probability density which gives the probability distribution law of the position variable of the electrons moving under the control by the potential $V(x)$.

2 Derivation of the Schrödinger equation

For the purpose of the natural statistical study on the phenomenon of the potential well, we solve the local variational problem by using the local variational principle on the basis of the laws of natural statistical physics. Thereby we derive the Schrödinger equation.

Then, the L^2-density $\psi(x,t)$ which determines the natural probability distribution of the position variable of the physical system at the time t is given by the following theorem.

Theorem 2.1 *Assume that the initial natural probability distribution of the position variable is given by the L^2-density $\psi(x)$.*

Then, the L^2-density $\psi(x,t)$ which determines the natural probability distribution of the position variable of the physical system is the solution of the initial value problem of the time evolving Schrödinger equation

$$i\hbar \frac{\partial \psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi(x,t)}{\partial x^2} + V(x)\psi(x,t),$$

$$\psi(x,0) = \psi(x), \text{(Initial Condition)},$$

$$(-\infty < x < \infty,\ 0 < t < \infty).$$

Then the phenomenon of the potential well is understood by solving the initial value problem of the Schrödinger equation. Namely, when the initial distribution of the electrons is given, the L^2-density $\psi(x,t)$ is varying accompanying the motion of the electrons. Then the expectation of a physical quantity calculated by using the L^2-density $\psi(x,t)$ varies. By virtue of the observation of the variation of such a physical quantity, we can understand the phenomenon of potential well.

By using this mathematical model, we can explain the motion of free electrons in a metallic conductor. Thereby we can explain that a metallic matter has the property of electric conductivity.

3 Eigenvalue problem for the Schrödinger operator

We have the solutions of the Schrödinger equation on the stationary state in the following theorem.

Theorem 3.1 *We have the L^2_{loc}-solutions $\psi_1^{(n)}(x)$, $\psi_2^{(n)}(x)$, $\psi_1^{(p)}(x)$, $\psi_2^{(p)}(x)$ of the Schrödinger equation on the stationary state*

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_j^{(p)}(x)}{dx^2} + V(x)\psi_j^{(p)}(x) = \mathcal{E}_p\psi_j^{(p)}(x),$$

$$(\mathcal{E}_p = \frac{p^2}{2m} - V, \text{ or } \mathcal{E}_p = \frac{p^2}{2m}),$$

$$(j = 1, 2\ ;\ -\infty < x < \infty,\ -\infty < p < \infty).$$

By solving the Schrödinger equation in the stationary state, we determine the system of eigenfunctions as the solutions of the eigenvalue problem for the Schrödinger operator.

We give the concrete expressions of the system of eigenfunctions $\psi_1^{(n)}(x)$, $\psi_2^{(n)}(x)$, $\psi_1^{(p)}(x)$, $\psi_2^{(p)}(x)$ in the following theorems 3.2∼ 3.4.

Theorem 3.2 *In the case* $0 > \mathcal{E} \geq -V$, *we put* $\mathcal{E} = \dfrac{p^2}{2m} - V$, $(|p| < \sqrt{2mV})$. *Then, we have the piecewise* C^1 *and continuous solutions* $\psi_1^{(p)}(x)$ *and* $\psi_2^{(p)}(x)$ *of the Schrödinger equation as follows*:

(i) *In the cases* $p = \left(n - \dfrac{1}{2}\right)\dfrac{h}{2a}$, $(n = 1, 2, \cdots, N)$, *we have*

$$\psi_1^{(p)}(x) = \psi_1^{(n)}(x) = \begin{cases} c\cos\left(n - \dfrac{1}{2}\right)\dfrac{\pi x}{a}, & (|x| \leq a), \\ 0, & (|x| > a). \end{cases}$$

(ii) *In the cases* $p = \dfrac{nh}{2a}$, $(n = 1, 2, \cdots, N)$, *we have*

$$\psi_2^{(p)}(x) = \psi_2^{(n)}(x) = \begin{cases} c\sin\dfrac{n\pi x}{a}, & (|x| \leq a), \\ 0, & (|x| > a). \end{cases}$$

Here, $c = \dfrac{1}{\sqrt{a}}$ *is the normalization constant. Further* N *is the biggest natural number among which satisfy the condition*

$$N < \frac{2a\sqrt{2mV}}{h}.$$

In the theorem 3.2, the interval to which the point $x = a$ in the domain of eigenfunction solutions belongs is different from the interval in the definition of potential well. As for this fact, it is enough to understand as follows.

In the definition of potential well, it is good that the point $x = a$ is the left end point of the interval $[a, \infty)$ by virtue of its physical meanings.

Nevertheless, the eigenfunction solution in the theorem 3.2 is L^2_{loc}-function.

Really, this solution is a piecewise C^1 and continuous function. Therefore, by the reason of the continuity condition of the solution at the point $x = a$, we can consider that the point $x = a$ is the right end point of the interval $[-a, a]$.

Here we express the domain of the function so that the symmetry property of the function is expressed.

Theorem 3.3 *In the case $\mathcal{E} \geq 0$, we put*

$$\mathcal{E} = \frac{p^2}{2m} = \frac{p'^2}{2m} - V, \ (|p'| \geq \sqrt{2mV}, \ -\infty < p < \infty).$$

Here, p' denotes a function of p.

Then we have the solution $\psi_1^{(p)}(x)$ of the Schrödinger equation as follows :

(i) *In the case $x > a$, we have*

$$\psi_1^{(p)}(x) = A_1(p)e^{ip(x-a)/\hbar} + B_1(p)e^{-ip(x-a)/\hbar}$$

$$= c(\cos p'a/\hbar \cdot \cos p(x-a)/\hbar - \frac{p'}{p} \sin p'a/\hbar \cdot \sin p(x-a)/\hbar).$$

(ii) *In the case $x < -a$, we have*

$$\psi_1^{(p)}(x) = A_1(p)e^{-ip(x+a)/\hbar} + B_1(p)e^{ip(x+a)/\hbar}$$

$$= c(\cos p'a/\hbar \cdot \cos p(x+a)/\hbar + \frac{p'}{p} \sin p'a/\hbar \cdot \sin p(x+a)/\hbar).$$

(iii) *In the case $|x| \leq a$, we have*

$$\psi_1^{(p)}(x) = c \cos p'x/\hbar.$$

Here, in the cases (i), (ii), *$A_1(p)$ and $B_1(p)$ are defined as follows :*

$$A_1(p) = \frac{c}{2p}(p \cos p'a/\hbar + ip' \sin p'a/\hbar),$$

$$B_1(p) = \frac{c}{2p}(p \cos p'a/\hbar - ip' \sin p'a/\hbar).$$

Here, $c = \dfrac{1}{\sqrt{a}}$ denotes the normalization constant.

Theorem 3.4 *Let $\mathcal{E}, p, p'$ be as the same as in Theorem* 3.3. *Then we have the solutions $\psi_2^{(p)}(x)$ of the Schrödinger equation as follows*:

(iv) *In the case $x > a$, we have the solutions*

$$\psi_2^{(p)}(x) = A_2(p)e^{ip(x-a)/\hbar} + B_2(p)e^{-ip(x-a)/\hbar}$$

$$= c(\sin p'a/\hbar \cdot \cos p(x-a)/\hbar + \frac{p'}{p} \cos p'a/\hbar \cdot \sin p(x-a)/\hbar).$$

(v) *In the case* $x < -a$*, we have the solutions*

$$\psi_2^{(p)}(x) = -A_2(p)e^{-ip(x+a)/\hbar} - B_2(p)e^{ip(x+a)/\hbar}$$

$$= -c(\sin p'a/\hbar \cdot \cos p(x+a)/\hbar - \frac{p'}{p}\cos p'a/\hbar \cdot \sin p(x+a)/\hbar).$$

(vi) *In the case* $|x| \leq a$*, we have the solutions*

$$\psi_2^{(p)}(x) = c \sin p'x/\hbar.$$

Here, we determine the constants $A_2(p)$*,* $B_2(p)$ *in the cases* (iv), (v) *as follows :*

$$A_2(p) = \frac{c}{2p}(p \sin p'a/\hbar - ip' \cos p'a/\hbar),$$

$$B_2(p) = \frac{c}{2p}(p \sin p'a/\hbar + ip' \cos p'a/\hbar).$$

Then, $c = \dfrac{1}{\sqrt{a}}$ *denotes the normalization constant.*

It is proved that this system of eigenfunctions satisfies the ortho-normalization conditions.

Theorem 3.5 *We have the ortho-normalization conditions in the following :*

(1) $\displaystyle\int_{-a}^{a} \psi_1^{(n')}(x)^* \psi_1^{(n)}(x)dx = \delta_{n'n}, \quad (n, n' = 1, 2, \cdots, N).$

(2) $\displaystyle\int_{-a}^{a} \psi_2^{(n')}(x)^* \psi_2^{(n)}(x)dx = \delta_{n'n}, \quad (n, n' = 1, 2, \cdots, N).$

(3) $\displaystyle\int_{-a}^{a} \psi_1^{(n)}(x)^* \psi_2^{(n')}(x)dx = \int_{-a}^{a} \psi_2^{(n')}(x)^* \psi_1^{(n)}(x)dx = 0,$

$(n, n' = 1, 2, \cdots, N).$

(4) $\displaystyle\int_{-\infty}^{\infty} \psi_j^{(q)}(x)^* \psi_j^{(p)}(x)dx = \delta(q-p),$

$(j = 1, 2, \ ; -\infty < p, q < \infty).$

(5) $\displaystyle\int_{-\infty}^{\infty} \psi_1^{(q)}(x)^* \psi_2^{(p)}(x)dx = \int_{-\infty}^{\infty} \psi_2^{(q)}(x)^* \psi_1^{(p)}(x)dx = 0,$

$(-\infty < p, q < \infty).$

(6) $\displaystyle\int_{-\infty}^{\infty} \psi_i^{(n)}(x)^* \psi_j^{(p)}(x)dx = \int_{-\infty}^{\infty} \psi_j^{(p)}(x)^* \psi_i^{(n)}(x)dx = 0,$

$(i = 1, 2, \ n = 1, 2, \cdots, N; \ j = 1, 2, \ -\infty < p < \infty).$

Here, we propose the theorem of the completeness of the system of eigenfunctions. But the proof of this theorem is an open problem.

Theorem 3.6 *We have the following equality :*

$$\sum_{n=1}^{N} \Big(\psi_1^{(n)}(x')^* \psi_1^{(n)}(x) + \psi_2^{(n)}(x')^* \psi_2^{(n)}(x) \Big)$$

$$+ \int_{-\infty}^{\infty} \Big(\psi_1^{(p)}(x')^* \psi_1^{(p)}(x) + \psi_2^{(p)}(x')^* \psi_2^{(p)}(x) \Big)\, dp$$

$$= \delta(x' - x),\ (-\infty < x, x' < \infty).$$

Then the theorem 3.6 is equivalent to the next corollary 3.1.

Corollary 3.1 *For the function $\psi(x) \in L^2$, we have the equality:*

$$\int_{-\infty}^{\infty} |\psi(x)|^2 dx = \sum_{n=1}^{N} \big(\, |a_n|^2 + |b_n|^2 \, \big)$$

$$+ \int_{-\infty}^{\infty} \big(\, |a(p)|^2 + |b(p)|^2 \, \big)\, dp.$$

Here, we put

$$a_n = \int_{-\infty}^{\infty} \psi_1^{(n)}(x)^* \psi(x) dx,$$

$$b_n = \int_{-\infty}^{\infty} \psi_2^{(n)}(x)^* \psi(x) dx,$$

$$a(p) = \int_{-\infty}^{\infty} \psi_1^{(p)}(x)^* \psi(x) dx,$$

$$b(p) = \int_{-\infty}^{\infty} \psi_2^{(p)}(x)^* \psi(x) dx.$$

Theorem 3.7 (Theorem of eigenfunction expansion) *When an initial natural probability distribution of the physical system is given by the L^2-density $\psi(x)$, we have the equality:*

$$\psi(x) = \sum_{n=1}^{N} \Big(a_n \psi_1^{(n)}(x) + b_n \psi_2^{(n)}(x) \Big)$$

$$+ \int_{-\infty}^{\infty} \Big(a(p) \psi_1^{(p)}(x) + b(p) \psi_2^{(p)}(x) \Big)\, dp.$$

Here we put

$$a_n = \int_{-\infty}^{\infty} \psi_1^{(n)}(x)^* \psi(x) dx,$$

$$b_n = \int_{-\infty}^{\infty} \psi_2^{(n)}(x)^* \psi(x) dx,$$

$$a(p) = \int_{-\infty}^{\infty} \psi_1^{(p)}(x)^* \psi(x) dx,$$

$$b(p) = \int_{-\infty}^{\infty} \psi_2^{(p)}(x)^* \psi(x) dx.$$

Then the expectation value $\overline{E}$ of the energy of this physical system is equal to the following:

$$\overline{E} = \sum_{n=1}^{N} \left\{ \left(n - \frac{1}{2} \right)^2 \frac{h^2}{8ma^2} |a_n|^2 + \frac{n^2 h^2}{8ma^2} |b_n|^2 \right\}$$

$$+ \int_{-\infty}^{\infty} \frac{p^2}{2m} (|a(p)|^2 + |b(p)|^2) dp.$$

Now we put

$$\psi_j^{(n)}(x,t) = \psi_j^{(n)}(x) \exp\left(-i \frac{\mathcal{E}_p}{\hbar} t \right),$$

$$(j = 1, 2,\ \mathcal{E}_p = \frac{p^2}{2m} - V)$$

$$\psi_k^{(p)}(x,t) = \psi_k^{(p)}(x) \exp\left(-i \frac{\mathcal{E}_p}{\hbar} t \right),$$

$$(k = 1, 2,\ \mathcal{E}_p = \frac{p^2}{2m}).$$

The solution $\psi(x,t)$ of the Schrödinger equation is given by the equality

$$\psi(x,t) = \sum_{n=1}^{N} \left(a_n \psi_1^{(n)}(x,t) + b_n \psi_2^{(n)}(x,t) \right)$$

$$+ \int_{-\infty}^{\infty} \left(a(p) \psi_1^{(p)}(x,t) + b(p) \psi_2^{(p)}(x,t) \right) dp.$$

Corollary 3.2 *We use the notations used in theorem* 3.7. *Assume that the initial distribution of the physical system is given by the following L^2-density*

$$\psi(x) = \sum_{n=1}^{N} \left(a_n \psi_1^{(n)}(x) + b_n \psi_2^{(n)}(x) \right).$$

Then the solution $\psi(x,t)$ of the time evolving Schrödinger equation of the physical system is given by the equality

$$\psi(x,t) = \sum_{n=1}^{N} \Big(a_n \psi_1^{(n)}(x,t) + b_n \psi_2^{(n)}(x,t) \Big)$$

Here the expectation value $\overline{E}$ of the energy of this physical system is given by the equality

$$\overline{E} = \sum_{n=1}^{N} \left\{ \Big(n - \frac{1}{2} \Big)^2 \frac{h^2}{8ma^2} |a_n|^2 + \frac{n^2 h^2}{8ma^2} |b_n|^2 \right\}.$$

4 Meanings of the phenomena and natural statistical phenomena

Now we study the structure of the physical system $\Omega = (\Omega, \mathcal{B}, P)$.

Really, the physical system Ω has the following structure in the stationary state.

Namely, Ω has the following direct sum decomposition:

$$\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4, \text{ (direct sum).} \tag{4.1}$$

$$\Omega_1 = \bigcup_{n=1}^{N} \Omega_{1n}, \ \Omega_2 = \bigcup_{n=1}^{N} \Omega_{2n}, \text{ (direct sum),} \tag{4.2}$$

$$\Omega_j = \bigcup_{-\infty<p<\infty} \Omega_{jp}, \ (j = 3,4), \text{ (direct sum),} \tag{4.3}$$

$$P(\Omega_{1n}) = |a_n|^2, \ P(\Omega_{2n}) = |b_n|^2,$$
$$P(\Omega_{3p}) = |a(p)|^2, \ P(\Omega_{4p}) = |b(p)|^2,$$
$$(-\infty < p < \infty).$$

Then, for $A \in \mathcal{B}$, we have the equality:

$$P(A) = \sum_{n=1}^{N} \Big(P(\Omega_{1n}) P_{\Omega_{1n}}(A) + P(\Omega_{2n}) P_{\Omega_{2n}}(A) \Big)$$
$$+ \int_{-\infty}^{\infty} \Big(P_3(A|p) P(\Omega_{3p}) + P_4(A|p) P(\Omega_{4p}) \Big)\, dp$$

$$= \sum_{n=1}^{N} \Big(\ |a_n|^2 P_{\Omega_{1n}}(A) + |b_n|^2 P_{\Omega_{2n}}(A) \ \Big)$$

$$+ \int_{-\infty}^{\in fty} \Big(\ P_3(A|p)|a(p)|^2 + P_4(A|p)|b(p)|^2 \ \Big) \ dp.$$

Here, $P_{\Omega_{1n}}(A), P_{\Omega_{2n}}(A), P_3(A|p)$ and $P_4(A|p)$ denote the conditional probabilities.

Then, for $j = 1, 2, \ n = 1, 2, \cdots, N$, we call $(\Omega_{jn}, \mathcal{B} \cap \Omega_{jn}, P_{\Omega_{jn}}(\cdot))$ a **proper physical system**, and for $j = 3, 4, \ -\infty < p < \infty$, we call $(\Omega_{jp}, \mathcal{B} \cap \Omega_{jp}, P_j(\cdot|p))$ a **generalized proper physical system**.

Then considering the formulas (4.1)∼(4.3) and calculations until now, and using the symbols in Ito [3], section 6.2 law II and section 6.5, law II′, we have the following, for j, n, p chosen as in the above :

For a measurable set A in $\boldsymbol{R}$ and a measurable set B in $\boldsymbol{R}$, we have the following :

$$P_{\Omega_{jn}}(\{\rho \in \Omega_{jn} : x(\rho) \in A\}) = \int_A |\psi_j^{(n)}(x)|^2 dx, \ (j = 1, 2),$$

$$P_{\Omega_{jn}}(\{\rho \in \Omega_{jn} : p(\rho) \in B\}) = \int_B |\hat{\psi}_j^{(n)}(p)|^2 dp, \ (j = 1, 2),$$

$$P(\{\rho \in \Omega_{jp}; x(\rho) \in A \cap S\}|p) = \frac{\displaystyle\int_{A \cap S} |\psi_{jS}^{(p)}(x)|^2 dx}{\displaystyle\int_S |\psi_{jS}^{(p)}(x)|^2 dx}, \ (j = 3, 4),$$

$$P(\{\rho \in \Omega_{jp}; p(\rho) \in B\}|p) = \frac{\displaystyle\int_B |\hat{\psi}_{jS}^{(p)}(q)|^2 dq}{\displaystyle\int_{-\infty}^{\infty} |\hat{\psi}_{jS}^{(p)}(q)|^2 dq}, \ (j = 3, 4),$$

$$P(\Omega_{jp}|p) = \lim_S P(\{\rho \in \Omega_{jp}; x(\rho) \in S\}) = 1, \ (j = 3, 4).$$

Here , $\lim_S$ denotes the Moore-Smith limit.

Therefore, the conditional expectation values $\overline{E}_{jn}$ of the proper physical system Ω_{jn} and $\overline{E}_{jp}$ of the generalized proper physical system Ω_{jp} are the following respectively:

$$\overline{E}_{jn} = E_{\Omega_{jn}} \left[\frac{1}{2m} p(\rho)^2 + V(x(\rho)) \right]$$

$$= \begin{cases} \left(n - \dfrac{1}{2}\right)^2 \dfrac{h^2}{8ma^2}, & (j = 1), \\ \dfrac{n^2 h^2}{8ma^2}, & (j = 2), \end{cases}$$

$$\overline{E}_{jp} = \lim_{n\to\infty} J_n[\psi^{(p)}_{j,n}] = \frac{p^2}{2m},$$
$$(j = 3, 4, ; -\infty < p < \infty).$$

Then, by virtue of the relation of the total physical system Ω and the proper physical systems Ω_{jn} and the generalized proper physical systems Ω_{jp}, the energy expectation value $\overline{E}$ of the total physical system is equal to the following:

$$\begin{aligned}\overline{E} &= E\left[\frac{1}{2m}p(\rho)^2 + V(x(\rho))\right] \\ &= \sum_{n=1}^{N}\Big(\ \overline{E}_{1n}|a_n|^2 + \overline{E}_{2n}|b_n|^2\ \Big) \\ &\quad + \int_{-\infty}^{\infty}\Big(\ \overline{E}_{3p}|a(p)|^2 + \overline{E}_{4p}|b(p)|^2\ \Big)\,dp \\ &= \sum_{n=1}^{N}\Big\{\ \left(n-\frac{1}{2}\right)^2\frac{h^2}{8ma^2}|a_n|^2 + \frac{n^2h^2}{8ma^2}|b_n|^2\ \Big\} \\ &\quad + \int_{-\infty}^{\infty}\frac{p^2}{2m}(|a(p)|^2 + |b(p)|^2)dp.\end{aligned}$$

Now, assume that the initial distribution of the physical system is given by the L^2-density in the corollary 3.2. Then, if we assume that the following condition are satisfied :

$$\psi(x) = 0,\ (|x| > a),$$

we have the equality

$$\psi(x,t) = 0,\ (|x| > a,\ 0 < t < \infty).$$

Therefore, when the system of electrons, whose energy expectation value $\mathcal{E}$ belongs to the interval $[-V, 0)$, are delivered in the region $|x| \le a$ at the initial time, we know that the probability of the possibility that these electrons run out of the potential region is equal to 0. .

Namely, the electrons of low energy trapped in the potential well are continuing to stay at the state where these electrons are trapped in the potential well as they are.

5 Meaning of the impact force

Here we study the meaning of the impact force in the section 1.
In the space $\boldsymbol{R}$, we consider the potential:

$$V_\varepsilon(x) = \begin{cases} 1, & (x \geq 0), \\ \dfrac{1}{\varepsilon}(x+\varepsilon), & (-\varepsilon \leq x < 0), \\ 0, & (x < -\varepsilon), \end{cases}$$

$$V(x) = \begin{cases} 1, & (x \geq 0), \\ 0, & (x < 0). \end{cases}$$

Then, in the wide sense of uniform convergence, we have the limit

$$V_\varepsilon(x) \longrightarrow V(x), \ (\varepsilon \longrightarrow +0)$$

Then, we have the equality

$$-\frac{dV_\varepsilon(x)}{dx} = \begin{cases} 0, & (x \geq 0), \\ -\dfrac{1}{\varepsilon}, & (-\varepsilon \leq x < 0), \\ 0, & (x < -\varepsilon) \end{cases} .$$

Further, in the sense of convergence as the density function or the Radon measure, we have the limit

$$-\frac{dV_\varepsilon(x)}{dx} \longrightarrow g(x), \ (\varepsilon \to +0).$$

Then, for $x \in \boldsymbol{R}$, we have the limit

$$V_\varepsilon(x) = \int_{-\infty}^{x} \frac{dV_\varepsilon(x)}{dx}$$

$$\longrightarrow V(x) = \int_{-\infty}^{x} g(x)dx = \begin{cases} 1, & (x \geq 0), \\ 0, & (x < 0) \end{cases} .$$

Here, because the density function $V(x)$ is $\delta(x)$, We have the equality $g(x) = \delta(x)$.

Therefore, $g(x) = \delta(x)$ is considered to be an impact force at the point $x = 0$. Then the $-$(work) by the impact force $\delta(x)$ is considered to be the potential energy $V(x)$.

Here we consider the meaning of the impact force. Now, if we put

$$F(x) = -\frac{dV(x)}{dx} = -\delta(x),$$

we have thw equalities

$$\int_{\{0\}} F(x)dx = -\int_{\{0\}} \delta(x)dx = -1,$$

$$\int_{|x|>0} F(x)dx = -\int_{|x|>0} \delta(x)dx = 0.$$

This means that the work by the force $-\delta(x)$ concentrates on $\{0\}$ and its work is equal to -1.

In this meaning, we can consider the force $-\delta(x)$ as the impact force at the point $x = 0$.

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