

On Existence of Global Solutions for Some Mildly Degenerate Nonlinear Kirchhoff Strings with Linear Dissipation

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Abstract

We study the existence of global solutions to the initial-boundary value problem for the degenerate nonlinear hyperbolic equation of Kirchhoff type with linear dissipation :

$$\frac{\partial^2 u}{\partial t^2} - \left(\int_a^b \left| \frac{\partial u}{\partial x} \right|^2 dx \right)^\gamma \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial t} + |u|^p u = 0.$$

In the case of $0 < \gamma < 1$, under the conditions that $p > 4\gamma$ and the size of initial data is suitably small, we derive the global existence theorem.

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Introduction

In this paper we consider the initial-boundary value problem for the degenerate hyperbolic equation of Kirchhoff type with linear dissipation :

$$u_{tt} - \|u_x(t)\|^{2\gamma} u_{xx} + u_t + |u|^p u = 0 \quad \text{in } (a, b) \times (0, \infty) \quad (0.1)$$

with the initial data and boundary conditions

$$\begin{aligned} u(x, 0) &= u_0(x), & u_t(x, 0) &= u_1(x), & a < x < b, \\ u(a, t) &= u(b, t) = 0, & 0 < t < \infty \end{aligned}$$

where $u = u(x, t)$ is an unknown real function and $a < b$ and

$$0 < \gamma < 1$$

and $p > 0$ and the symbol $\|\cdot\|$ means the usual norm of $L^2 = L^2(a, b)$.

Equation (0.1) describes small amplitude vibrations of an elastic stretched string and was introduced by Kirchhoff [4].

When $\gamma \geq 1$, in previous paper [9], we have proved the existence of global solutions under the condition that the size of the initial data is small (cf. [5], [8], [10], [12]).

Our interest in this paper is the problem in the case of $0 < \gamma < 1$ (cf. [3], [11] for equations without the nonlinear term $|u|^p u$). In order to get an a-priori estimate for H^2 -norm of the solution $u(t)$, we derive the estimate for $\|u_{xx}(t)\|^2/\|u_x(t)\|^2$, and we give the decay estimate $\|u(t)\|_{H^2}^2 \leq C(1+t)^{-\frac{1}{\gamma}}$. Our main result is Theorem 3.3 in section 3.

The notations we use in this paper are standard. The symbol (\cdot, \cdot) means the inner product in L^2 or sometimes duality between the space X and its dual X' , and the norm of L^p is often written as $\|\cdot\|_p$ ($\|\cdot\| = \|\cdot\|_2$ for $p = 2$) simplicity. We put $(a)^+ = \max(0, a)$ where $1/(a)^+ = \infty$ if $(a)^+ = 0$. The constant c_* is the Sobolev-Poincaré constant, that is, for $1 \leq p \leq \infty$,

$$\|v\|_p \leq c_* \|v_x\|. \quad (0.2)$$

Positive constants will be denoted by C and will change line to line.

1 Preliminaries

By applying the Banach contraction mapping theorem, we get the following local existence theorem (see [1], [2], [9] and the references cited therein).

Proposition 1.1 *Suppose that the initial data $\{u_0, u_1\}$ belong to $H^2 \cap H_0^1 \times H_0^1$ and $u_0 \neq 0$. Then, the problem (0.1) admits a unique local solution $u(t)$ in the class $C([0, T]; H^2 \cap H_0^1) \cap C^1([0, T]; H_0^1) \cap C^2([0, T]; L^2)$ for some $T \equiv T(\|u_0\|_{H^2}, \|u_1\|_{H^1}) > 0$. Moreover, if $\|u_x(t)\| > 0$ for $0 \leq t < T$, at least one of the following statements is valid*

- (i) $T = \infty$;
- (ii) $\|u(t)\|_{H^2} + \|u_t(t)\|_{H^1} \rightarrow \infty$ as $t \rightarrow T^-$;
- (iii) $\|u_x(t)\| \rightarrow 0$ as $t \rightarrow T^-$.

We define the energy and the potential associated with (0.1) by

$$E(u, u_t) \equiv \frac{1}{2} \|u_t\|^2 + J(u) \quad (1.1)$$

and

$$J(u) \equiv \frac{1}{2(\gamma+1)} \|u_x\|^{2(\gamma+1)} + \frac{1}{p+2} \|u\|_{p+2}^{p+2}, \quad (1.2)$$

respectively.

In what follows, we denote

$$M(t) \equiv \|u_x(t)\|^2 \quad \text{and} \quad E(t) \equiv E(u(t), u_t(t)) \quad (1.3)$$

($E(0) \equiv E(u_0, u_1)$ for $t = 0$) and $J(t) \equiv J(u(t))$ for simplicity.

Moreover, we introduce the function $H(t)$ (i.e. modified second energy) by

$$H(t) \equiv \frac{\|u_{xt}(t)\|^2}{M(t)} + \frac{\|u_{xx}(t)\|^2}{M(t)^{1-\gamma}}. \quad (1.4)$$

2 Energy Estimate

The energy $E(t) \equiv E(u(t), u_t(t))$ given by (1.1) has the energy identity

$$\frac{d}{dt} E(t) + \|u_t(t)\|^2 = 0 \quad (2.1)$$

or

$$E(t) + \int_0^t \|u_t(s)\|^2 ds = E(0). \quad (2.2)$$

Indeed, multiplying (0.1) by u_t and integrating it over (a, b) or $(a, b) \times (0, t)$, we have (2.1) or (2.2). Moreover, applying energy method together with the Nakao inequality (see [6], [7]), we have the following the decay estimate of the energy $E(t)$ (see [6], [10], [11] for the proof).

Proposition 2.1 *Let $u(t)$ be a solution of (0.1).*

Then, the energy $E(t)$ satisfies

$$E(t) \leq \left(E(0)^{-\frac{\gamma}{\gamma+1}} + d_1^{-1}(t-1)^+ \right)^{-\frac{\gamma+1}{\gamma}}, \quad (2.3)$$

where we define

$$d_1 \equiv \left(3E(0)^{\frac{\gamma}{2(\gamma+1)}} + 20c_* \right)^2. \quad (2.4)$$

Proof. Integrating (2.1) over $[t, t+1]$, we have

$$\int_t^{t+1} \|u_t(s)\|^2 ds = E(t) - E(t+1) \quad (\equiv D(t)^2). \quad (2.5)$$

Then, there exist two numbers $t_1 \in [t, t+1/4]$ and $t_2 \in [t+3/4, t+1]$ such that

$$\|u_t(t_j)\|^2 \leq 4D(t)^2 \quad \text{for } j = 1, 2. \quad (2.6)$$

Multiplying (0.1) by u and integrating it over (a, b) , we have

$$M(t)^{\gamma+1} + \|u\|_{p+2}^{p+2} = \|u_t\|^2 - \frac{d}{dt}(u_t, u) - (u_t, u),$$

and integrating the resulting equality over $[t_1, t_2]$, we observe from (2.5) that

$$\begin{aligned} & \int_{t_1}^{t_2} \left(M(s)^{\gamma+1} + \|u(s)\|_{p+2}^{p+2} \right) ds \\ & \leq \int_t^{t+1} \|u_t(s)\|^2 ds + \sum_{j=1}^2 \|u_t(t_j)\| \|u(t_j)\| + \int_t^{t+1} \|u_t(s)\| \|u(s)\| ds \\ & \leq D(t)^2 + 5c_* D(t) \sup_{t \leq s \leq t+1} M(s)^{\frac{1}{2}} \\ & \leq D(t)^2 + 5c_* D(t) (2(\gamma+1)E(t))^{\frac{1}{2(\gamma+1)}} \end{aligned} \quad (2.7)$$

where we used the fact that $E(t)$ is a non-increasing function at the last inequality.

Integrating (2.1) over $[t, t_2]$, we have

$$\begin{aligned} E(t) &= E(t_2) + \int_t^{t_2} \|u_t(s)\|^2 ds \\ &\leq 2 \int_{t_1}^{t_2} E(s) ds + \int_t^{t+1} \|u_t(s)\|^2 ds \\ &\leq 2 \int_t^{t+1} \|u_t(s)\|^2 ds + \int_{t_1}^{t_2} \left(\frac{1}{\gamma+1} M(s)^{\gamma+1} + \frac{2}{p+2} \|u(s)\|_{p+2}^{p+2} \right) ds \\ &\leq 3D(t)^2 + 5c_* D(t) (2(\gamma+1)E(t))^{\frac{1}{2(\gamma+1)}}, \end{aligned}$$

where we used (2.7) at the last inequality. Moreover, since $D(t)^2 \leq E(0)^{\frac{\gamma}{\gamma+1}} E(t)^{\frac{1}{\gamma+1}}$, we observe

$$E(t) \leq \left(3E(0)^{\frac{1}{\gamma+1}} + 20c_* \right) D(t) E(t)^{\frac{1}{2(\gamma+1)}}$$

and the Young inequality yields

$$E(t) \leq \left(\left(3E(0)^{\frac{1}{\gamma+1}} + 20c_* \right) D(t) \right)^{\frac{2(\gamma+1)}{2\gamma+1}}$$

or

$$\begin{aligned} E(t)^{1+\frac{\gamma}{\gamma+1}} &= E(t)^{\frac{2\gamma+1}{\gamma+1}} \leq d_1^2 D(t)^2 \\ &\leq d_1^2 (E(t) - E(t+1)). \end{aligned} \quad (2.8)$$

Thus, applying the Nakao inequality to (2.8), we obtain the desired estimate (2.3). \square

Immediately, we obtain the following estimate as a corollary of Proposition 2.1.

Corollary 2.2 *If $q > \gamma$, under the assumption of Proposition 2.1, it holds that*

$$\int_0^t M(s)^q ds \leq B_q(0), \quad (2.9)$$

where we define

$$B_q(0) \equiv (2(\gamma + 1))^{\frac{q}{\gamma+1}} \left(E(0)^{\frac{q}{\gamma+1}} + \frac{\gamma d_1}{q - \gamma} E(0)^{\frac{q-\gamma}{\gamma+1}} \right). \quad (2.10)$$

Proof. From (1.1) and (1.2), we see

$$M(t)^{\gamma+1} \leq 2(\gamma + 1)J(t) \leq 2(\gamma + 1)E(t),$$

and then, we observe from (2.3) that

$$\begin{aligned} \int_0^t M(s)^q ds &\leq \int_0^t (2(\gamma + 1)E(s))^{\frac{q}{\gamma+1}} ds \\ &\leq (2(\gamma + 1))^{\frac{q}{\gamma+1}} \int_0^t \left(E(0)^{-\frac{\gamma}{\gamma+1}} + d_1^{-1}(s-1)^+ \right)^{-\frac{q}{\gamma}} ds, \end{aligned}$$

which gives the desired estimate (2.9) if $q > \gamma$. \square

3 A-priori Estimate

Proposition 3.1 *Let $u(t)$ be a solution of (0.1). In addition to the assumption of Proposition 2.1, suppose that $p > 4\gamma$ and*

$$M(t) > 0 \quad \text{and} \quad (\gamma + 2)^2 H(t) \leq 1. \quad (3.1)$$

Then, it holds that

$$F(t) \equiv \frac{\|u_{xx}(t)\|^2}{M(t)} + Q(t) \leq d_2, \quad (3.2)$$

where we define

$$Q(t) \equiv \frac{1}{M(t)^{\gamma+1}} \left(M(t) \|u_{xt}(t)\|^2 - \left(\frac{1}{2} M'(t) \right)^2 \right) \geq 0, \quad (3.3)$$

$$d_2 \equiv F(0) + 2(p+1)c_*^p B_{\frac{p}{2}-\gamma}(0). \quad (3.4)$$

Proof. Since we observe from (0.1) that

$$\begin{aligned} M(t)^\gamma \frac{d}{dt} (\|u_{xx}\|^2) &= -2\|u_{xt}\|^2 - 2(u_{xtt}, u_{xt}) + 2(|u|^p u, u_{xxt}), \\ M(t)^\gamma \|u_{xx}\|^2 &= -\frac{1}{2}M'(t) + \left(\|u_{xt}\|^2 + \frac{1}{2}M''(t) \right) + (|u|^p u, u_{xx}), \end{aligned}$$

we have

$$\begin{aligned} \frac{d}{dt} \left(\frac{\|u_{xx}\|^2}{M(t)} \right) &= \frac{1}{M(t)^{\gamma+2}} (M(t)^\gamma (\|u_{xx}\|^2)' M(t) - M(t)^\gamma \|u_{xx}\|^2 M'(t)) \\ &= -2Q(t) - R(t) + S(t), \end{aligned} \quad (3.5)$$

where $Q(t)$ is defined by (3.3) and

$$\begin{aligned} R(t) &\equiv \frac{1}{M(t)^{\gamma+2}} \left(2M(t)(u_{xtt}, u_{xt}) + \left(M'(t)\|u_{xt}\|^2 - \frac{1}{2}M'(t)M''(t) \right) \right), \\ S(t) &\equiv \frac{1}{M(t)^{\gamma+2}} (2M(t)(f(u), u_{xxt}) - M'(t)(f(u), u_{xx})). \end{aligned}$$

On the other hand, we observe

$$\begin{aligned} \frac{d}{dt} Q(t) &= -(\gamma+2) \frac{M'(t)}{M(t)} \frac{1}{M(t)^{\gamma+2}} \left(M(t)\|u_{xt}\|^2 - \left(\frac{1}{2}M'(t) \right)^2 \right) \\ &\quad + \frac{1}{M(t)^{\gamma+2}} \left(2M(t)(u_{xtt}, u_{xt}) + M'(t)\|u_{xt}\|^2 - \frac{1}{2}M'(t)M''(t) \right) \\ &= -(\gamma+2) \frac{M'(t)}{M(t)} Q(t) + R(t). \end{aligned} \quad (3.6)$$

Adding (3.5) and (3.6), we have

$$\frac{d}{dt} \left(\frac{\|u_{xx}\|^2}{M(t)} + Q(t) \right) = -2 \left(1 + \frac{\gamma+2}{2} \frac{M'(t)}{M(t)} \right) Q(t) + S(t). \quad (3.7)$$

Here, we observe from (3.1) that

$$\frac{\gamma+2}{2} \frac{M'(t)}{M(t)} \leq (\gamma+2)H(t)^{\frac{1}{2}} \leq 1 \quad (3.8)$$

and

$$\begin{aligned} |S(t)| &\leq \frac{4(p+1)}{M(t)^{\gamma+2}} \|u\|_\infty^p \|u_x\|^3 \|u_{xt}\| \\ &\leq 4(p+1)c_*^p M(t)^{\frac{p}{2}-\gamma} \left(\frac{\|u_{xt}\|^2}{M(t)} \right)^{\frac{1}{2}} \\ &\leq 2(p+1)c_*^p M(t)^{\frac{p}{2}-\gamma} \end{aligned} \quad (3.9)$$

where we used (1.4) and (3.1) at the last inequality. Thus, we have from (3.7)–(3.9) that

$$\frac{d}{dt} \left(\frac{\|u_{xx}\|^2}{M(t)} + Q(t) \right) \leq 2(p+1)c_*^p M(t)^{\frac{p}{2}-\gamma}, \quad (3.10)$$

and then, integrating (3.10) and using (2.9), we obtain the desired estimate (3.2) if $p/2 - \gamma > \gamma$ (i.e. $p > 4\gamma$). \square

Proposition 3.2 *Let $u(t)$ be a solution of (0.1). Suppose that the assumption of Proposition 3.1 is fulfilled. Then, the function $H(t)$ given by (1.4) satisfies*

$$H(t) \leq H(0) + I(0), \quad (3.11)$$

where we define

$$I(0) \equiv 2(1-\gamma)^2 d_2^2 B_{2\gamma}(0) + 2c_*^{2p}(p+1)^2 B_p(0). \quad (3.12)$$

Proof. Multiplying (0.1) by $(-2u_{xxt}/M(t))$ and integrating it over (a, b) , we have

$$\begin{aligned} \frac{d}{dt} H(t) + \left(2 + \frac{M'(t)}{M(t)} \right) \frac{\|u_{xt}\|^2}{M(t)} &= -(1-\gamma) \frac{M'(t)}{M(t)^{2-\gamma}} \|u_{xx}\|^2 - 2 \frac{(|u|^p u, u_{xxt})}{M(t)} \\ &\equiv I_1 + I_2. \end{aligned}$$

We observe from (3.2) that

$$\begin{aligned} I_1 &\leq 2(1-\gamma) \frac{\|u_{xx}\|^2}{M(t)} M(t)^\gamma \frac{\|u_{xt}\|}{\|u_x\|} \leq 2(1-\gamma) d_2 M(t)^\gamma \left(\frac{\|u_{xt}\|^2}{M(t)} \right)^{\frac{1}{2}}, \\ I_2 &\leq \frac{2(p+1)}{M(t)} \|u\|_\infty^p \|u_x\| \|u_{xt}\| \leq 2c_*^p (p+1) M(t)^{\frac{p}{2}} \left(\frac{\|u_{xt}\|^2}{M(t)} \right)^{\frac{1}{2}}, \end{aligned}$$

and from (3.1) that

$$2 + \frac{M'(t)}{M(t)} \geq 2 - 2H(t)^{\frac{1}{2}} \geq 2 \frac{\gamma+1}{\gamma+2}.$$

Thus we have

$$\frac{d}{dt} H(t) + 2 \frac{\gamma+1}{\gamma+2} \frac{\|u_{xt}\|^2}{M(t)} \leq \left(2(1-\gamma) d_2 M(t)^\gamma + 2c_*^p (p+1) M(t)^{\frac{p}{2}} \right) \left(\frac{\|u_{xt}\|^2}{M(t)} \right)^{\frac{1}{2}}$$

and from the Young inequality,

$$\frac{d}{dt} H(t) \leq 2(1-\gamma)^2 d_2^2 M(t)^{2\gamma} + 2c_*^{2p} (p+1)^2 M(t)^p. \quad (3.13)$$

Therefore, integrating (3.13) and using (2.9), we obtain the desired estimate (3.11). \square

Our main result is as follows.

Theorem 3.3 *Let the initial data $\{u_0, u_1\}$ belong to $H^2 \cap H_0^1 \times H_0^1$ and $u_0 \neq 0$. Suppose that $p > 4\gamma$ and*

$$(\gamma + 1)^2(H(0) + I(0)) < 1, \quad (3.14)$$

where c_* , $E(0)$, $H(0)$, $I(0)$ are defined by (0.2), (1.3), (1.4), (3.12), respectively. Then, the problem (0.1) admits a unique global solution $u(t)$ in the class $C^0([0, \infty); H^2 \cap H_0^1) \cap C^1([0, \infty); H_0^1) \cap C^2([0, \infty); L^2)$ satisfying

$$\|u(t)\|_{H^2}^2 \leq C(1+t)^{-\frac{1}{\gamma}}, \quad (3.15)$$

and the energy satisfies

$$E(t) \equiv E(u(t), u_t(t)) \leq C(1+t)^{-1-\frac{1}{\gamma}}. \quad (3.16)$$

Proof. Let $u(t)$ be a solution of (0.1) on $[0, T_1)$. Since $M(0) > 0$ (by $u_0 \neq 0$), putting

$$T_2 \equiv \sup \{t \in [0, \infty) \mid M(s) > 0 \text{ for } 0 \leq s < t\},$$

we see that $T_2 > 0$ and $M(t) > 0$ for $0 \leq t < T_2$.

If $T_2 < T_1$, then we see $M(T_2) = 0$.

For $0 \leq t < T_2$, if we assume $(\gamma + 2)(H(0) + I(0))^{\frac{1}{2}} < 1$, then there exists $T_3 > 0$ such that

$$(\gamma + 2)H(t)^{\frac{1}{2}} \leq 1 \quad \text{for } 0 \leq t \leq T_3,$$

and hence, from Proposition 3.2,

$$H(t) \leq H(0) + I(0) \quad \text{for } 0 \leq t \leq T_3. \quad (3.17)$$

Thus, we observe from (3.17) that

$$(\gamma + 2)H(t)^{\frac{1}{2}} \leq (\gamma + 2)(H(0) + I(0))^{\frac{1}{2}} < 1 \quad \text{for } 0 \leq t \leq T_3,$$

and hence, we see $T_3 \geq T_2$, that is,

$$H(t) \leq \frac{1}{(\gamma + 2)^2} \quad \text{for } 0 \leq t < T_2. \quad (3.18)$$

Since $M(T_2) = 0$, we see from (3.18) that $\lim_{t \rightarrow T_2} E(t) = \lim_{t \rightarrow T_2} E(u(t), u_t(t)) = 0$.

We perform the change of variable $s = T_2 - t$ or $t = T_2 - s$, then the function $U(x, s) = u(x, T_2 - t)$ on $[0, T_2]$ satisfies that

$$U_{ss} - \|U_x(s)\|^{2\gamma} U_{xx} - U_s + |U|^p U = 0. \quad (3.19)$$

Multiplying (3.19) by U_s and integrating it over (a, b) , we have

$$\frac{d}{dt}E(U, U_s) = \|U_s\|^2 \leq 2E(U, U_s)$$

and from $E(U(0), U_s(0)) = \lim_{t \rightarrow T_2} E(u(t), u_t(t)) = 0$, we observe

$$E(U(s), U_s(s)) \leq 2 \int_0^s E(U(\tau), U_s(\tau)) d\tau.$$

Applying the Gronwall inequality, we have that $E(U(s), U_s(s)) = 0$ for $0 \leq s \leq T_2$ or $E(u(t), u_t(t)) = 0$ for $0 \leq t \leq T_2$ which contradicts $E(u_0, u_1) \equiv E(0) \geq J(0) \geq \frac{1}{2(\gamma+1)}M(0)^{\gamma+1} > 0$, and hence, we see $T_2 \geq T_1$ and $M(t) > 0$ for $0 \leq t \leq T_1$.

Thus, we conclude from (3.18) that $\|u(t)\|_{H^2} + \|u_t(t)\|_{H^1} < \infty$ for $t \geq 0$. Therefore, the local solution $u(t)$ of (0.1) in the sense of Proposition 1.1 can be continued globally in time. Also, from Proposition 2.1 and Proposition 3.1 we obtain the decay estimates (3.15) and (3.16). \square

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