On Asymptotic Behavior for Degenerate Hyperbolic Equations with Weak Dissipation

By

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Abstract

We consider the initial-boundary value problem for the degenerate hyperbolic equations with weak dissipation :
$\rho u'' + \left(\int_\Omega |A^{1/2}u(x,t)|^2\,dx\right) Au + a(t)u' = 0$. When the coefficient ρ or the initial data are small, we show the global existence theorem by using several identities. Moreover, we derive decay estimates of the solutions.

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1 Introduction and Results

In this paper we investigate on the global existence and asymptotic behavior of solutions to the initial-boundary value problem for the following degenerate hyperbolic equation with weak dissipation

$$\begin{cases} \rho u'' + \left(\int_\Omega |A^{1/2}u(x,t)|^2\,dx\right) Au + a(t)u' = 0 \quad \text{in} \quad \Omega \times (0,\infty)\,, \\ u(x,0) = u_0(x) \quad \text{and} \quad u'(x,0) = u_1(x) \\ u(x,t)\,\big|_{\partial\Omega} = 0\,, \end{cases} \tag{1.1}$$

where $u = u(x,t)$ is an unknown real value function, Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $' = \partial/\partial t$, $A = -\Delta = -\sum_{j=1}^N \partial^2/\partial x_j^2$ is the Laplace operator with the domain $\mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega)$, and $\rho > 0$ is a positive constant,

$$a(t) \equiv (1+t)^{-p}\,, \quad p > 0\,. \tag{1.2}$$

We assume that $0 < \rho \leq 1$ for simplicity.

When the dimension N is one, it is well-known that (1.1) describes small amplitude vibrations of an elastic stretched string, and (1.1) without any dissipative terms was introduced by Kirchhoff [4] (also see [3], [5]).

The local existence problem in Sobolev spaces has been already studied by many authors (e.g. [1], [2], [11] and the references cited therein).

On the other hand, in previous paper [8] we have proved the global existence problem of (1.1), under the condition that the initial data are small, and moreover, we have derived upper decay estimates of the solutions.

The purpose of this paper is to derive the global existence theorem for (1.1), when the coefficient ρ or the initial data are small. Moreover, we derive lower decay estimates of the solutions of (1.1) under the same assumption for ρ and the initial data (see [6], [9], and the references cited therein for $a(t) \equiv 1$).

The notations we use in this paper are standard. The symbol $\|\cdot\|$ is the norm in $L^2(\Omega)$ and the symbol $(\cdot, \cdot)$ means the inner product in $L^2(\Omega)$ or sometimes duality between the space X and its dual X'. We denote $[a]^+ = \max\{0\,, a\}$. Positive constants will be denoted by C and will change from line to line.

2 A-priori estimate

By applying the Banach contraction mapping theorem to the problem (1.1), we obtain the following local existence theorem (see [1], [2], [7], [10] for the proof).

Proposition 2.1 *Let the initial data (u_0, u_1) belong to $\mathcal{D}(A) \times \mathcal{D}(A^{1/2})$. Suppose that $u_0 \neq 0$. Then the problem (1.1) admits a unique local solution $u(t)$ in the class $C^0([0,T);\mathcal{D}(A)) \cap C^1([0,T);\mathcal{D}(A^{1/2})) \cap C^2([0,T);L^2(\Omega))$ for some $T = T(\|Au_0\|, \|A^{1/2}u_1\|) > 0$. Moreover, if $\|A^{1/2}u(t)\| > 0$ and $\|Au(t)\| + \|A^{1/2}u'(t)\| < \infty$ for $t \geq 0$, then we can take $T = \infty$.*

We denote $M(t) \equiv \|A^{1/2}u(t)\|^2$ for simplicity of the notation.

In what follows, let $u(t)$ be a solution of (1.1) and we assume that $M(t) > 0$.

By simple calculations, we see that the solution $u(t)$ satisfies the following identities associated with (1.1).

Proposition 2.2 *Let $k \geq 0$. The solution $u(t)$ satisfies that*

$$\frac{d}{dt}\left(\rho\frac{\|u'(t)\|^2}{M(t)^{1+k}} + \frac{M(t)}{M(t)^k}\right) + 2\left(a(t) + \frac{1+k}{2}\rho\frac{M'(t)}{M(t)}\right)\frac{\|u'(t)\|^2}{M(t)^{1+k}} = -k\frac{M'(t)}{M(t)^k}\,, \tag{2.1}$$

$$\frac{d}{dt}\left(\rho\frac{\|A^{1/2}u'(t)\|^2}{M(t)^{1+k}} + \frac{\|Au(t)\|^2}{M(t)^k}\right) + 2\left(a(t) + \frac{1+k}{2}\rho\frac{M'(t)}{M(t)}\right)\frac{\|A^{1/2}u'(t)\|^2}{M(t)^{1+k}}$$
$$= -k\frac{M'(t)}{M(t)^{k+1}}\|Au(t)\|^2\,. \tag{2.2}$$

Proof. Multiplying (1.1) by $2u'(t)$ (resp. $2Au(t)$) and $M(t)^{-1-k}$, and integrating it over Ω, we have the identity (2.1) (resp. (2.2)). $\square$

In order to get the a-priori estimate for (1.1), we introduce the function $H(t)$ by

$$H(t) \equiv \rho \frac{\|A^{1/2}u'(t)\|^2}{M(t)} + \|Au(t)\|^2 \,. \tag{2.3}$$

Then, we observe that

$$\rho \frac{|M'(t)|}{M(t)} \le 2\rho \frac{\|A^{1/2}u'(t)\|}{M(t)^{\frac{1}{2}}} \le 2(\rho H(t))^{\frac{1}{2}} \,. \tag{2.4}$$

Proposition 2.3 *Suppose that*

$$4(\rho H(t))^{\frac{1}{2}} \le a(t) \,. \tag{2.5}$$

Then, it holds that

$$\frac{\|Au(t)\|^2}{M(t)} \le C \tag{2.6}$$

where C is some positive constant.

Proof. From Equation (1.1) we observe

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\|Au(t)\|^2}{M(t)}\right) &= \frac{1}{M(t)^3}\left(2M(t)(M(t)Au, Au') - M'(t)(M(t)Au, Au)\right) \\
&= \frac{1}{M(t)^3}\Big(-2a(t)M(t)\|A^{1/2}u'(t)\|^2 - 2\rho M(t)(u'', Au') \\
&\quad + a(t)M'(t)(A^{1/2}u, A^{1/2}u') + \rho M'(t)(u'', Au)\Big) \\
&= -2a(t)Q(t) + \frac{\rho}{M(t)^3}(u'', -2M(t)Au' + M'(t)Au) \,,
\end{aligned} \tag{2.7}$$

where we define $Q(t)$ by

$$Q(t) \equiv \frac{1}{M(t)^3}\left(M(t)\|A^{1/2}u'(t)\|^2 - \left(\frac{1}{2}M'(t)\right)^2\right)$$

and we see $Q(t) \ge 0$. Then, we observe that

$$\begin{aligned}
\frac{d}{dt}Q(t) = -3\frac{M'(t)}{M(t)}Q(t) &+ \frac{1}{M(t)^3}\Big(M'(t)\|A^{1/2}u'(t)\|^2 + 2M(t)(u'', Au') \\
&- M'(t)\left((u'', Au) + \|A^{1/2}u'(t)\|^2\right)\Big) \\
= -3\frac{M'(t)}{M(t)}Q(t) &- \frac{1}{M(t)^3}(u'', -2M(t)Au' + M'(t)Au) \,.
\end{aligned} \tag{2.8}$$

Adding (2.7) to (2.8) $\times \rho$, we obtain

$$\frac{d}{dt}\left(\frac{\|Au(t)\|^2}{M(t)}+\rho Q(t)\right)=-2\left(a(t)+\frac{3}{2}\rho\frac{M'(t)}{M(t)}\right)Q(t)\,.$$

From (2.4), (2.5), and $Q(t)\geq 0$, we have

$$\frac{d}{dt}\left(\frac{\|Au(t)\|^2}{M(t)}+\rho Q(t)\right)\leq -\frac{1}{2}a(t)Q(t)\leq 0\,,$$

and hence, we obtain the desired estimate (2.6). □

Proposition 2.4 *Under the assumption of Proposition 2.3, it holds that*

$$\|A^{1/2}u(t)\|^2\geq C'(1+t)^{-(1+p)} \tag{2.9}$$

where C' is some positive constant.

Proof. From the identity (2.1) with $k=2$, we have that

$$\frac{d}{dt}\left(\rho\frac{\|u'(t)\|^2}{M(t)^3}+\frac{1}{M(t)}\right)+2\left(a(t)+\frac{3}{2}\rho\frac{M'(t)}{M(t)}\right)\frac{\|u'(t)\|^2}{M(t)^3}=-2\frac{M'(t)}{M(t)^2}\,,$$

and from (2.4) and (2.5) that

$$\begin{aligned}\frac{d}{dt}\left(\rho\frac{\|u'(t)\|^2}{M(t)^3}+\frac{1}{M(t)}\right)+\frac{1}{2}a(t)\frac{\|u'(t)\|^2}{M(t)^3}\\ \leq 4\left(a(t)\frac{\|u'(t)\|^2}{M(t)^3}\right)^{\frac{1}{2}}\left(\frac{1}{a(t)}\frac{\|Au(t)\|^2}{M(t)}\right)^{\frac{1}{2}}.\end{aligned}$$

The Young inequality yields

$$\frac{d}{dt}\left(\rho\frac{\|u'(t)\|^2}{M(t)^3}+\frac{1}{M(t)}\right)\leq C\frac{1}{a(t)}\frac{\|Au(t)\|^2}{M(t)}\leq C(1+t)^p\,, \tag{2.10}$$

where we used the estimate (2.6) at the last inequality. Integrating (2.10) in time, we obtain

$$\rho\frac{\|u'(t)\|^2}{M(t)^3}+\frac{1}{M(t)}\leq C(1+t)^{1+p}\,,$$

which gives the desired estimate (2.9). □

Proposition 2.5 *Under the assumption of Proposition 2.3, it holds that*

$$H(t)\leq H(0)\qquad \textit{for}\quad t\geq 0 \tag{2.11}$$

and

$$H(t) \le \left(H(0)^{-1} + k^{-1}(1-p)^{-1}\left((1+t)^{1-p} - 2^{1-p}\right)\right)^{-1} \quad \textit{for} \quad t \ge 1 \quad (2.12)$$

with $k = 2^{p+7}(H(0)+1)$, *that is,*

$$\|Au(t)\|^2 \le C(1+t)^{-(1-p)} \quad \textit{and} \quad \|A^{1/2}u'(t)\|^2 \le C(1+t)^{-2(1-p)}$$

where C *is some positive constant.*

Proof. From the identity (2.2) with $k = 0$, we have

$$\frac{d}{dt}H(t) + 2\left(a(t) + \frac{\rho}{2}\frac{M'(t)}{M(t)}\right)\frac{\|A^{1/2}u'(t)\|^2}{M(t)} = 0\,. \qquad (2.13)$$

We observe from (2.4) and (2.5) that

$$\frac{d}{dt}H(t) + a(t)\frac{\|A^{1/2}u'(t)\|^2}{M(t)} \le 0\,, \qquad (2.14)$$

and

$$H(t) \le H(0) \quad \text{or} \quad \frac{\|A^{1/2}u'(t)\|^2}{M(t)} \le \rho^{-1}H(t) \le \rho^{-1}H(0)\,. \qquad (2.15)$$

Next, we prove (2.12). Integrating (2.14) over $[t, t+1]$, we obtain

$$\int_t^{t+1} a(s)\frac{\|A^{1/2}u'(s)\|^2}{M(s)}\,ds \le H(t) - H(t+1) \quad \left(\equiv a(t+1)D(t)^2\right) \qquad (2.16)$$

or

$$\int_t^{t+1} \frac{\|A^{1/2}u'(s)\|^2}{M(s)}\,ds \le D(t)^2\,. \qquad (2.17)$$

Then, there exists two numbers $t_1 \in [t, t+1/4]$ and $t_2 \in [t+3/4, t+1]$ such that

$$\frac{\|A^{1/2}u'(t_j)\|^2}{M(t_j)} \le 4D(t)^2 \qquad \text{for} \quad j = 1, 2\,. \qquad (2.18)$$

On the other hand, multiplying (1.1) by Au and $M(t)^{-1}$, and integrating it over Ω, we have

$$\|Au(t)\|^2 + \frac{\rho}{2}\left(\frac{M'(t)}{M(t)}\right)^2 = \rho\frac{\|A^{1/2}u'(t)\|^2}{M(t)} - \frac{\rho}{2}\frac{d}{dt}\frac{M'(t)}{M(t)} - \frac{1}{2}a(t)\frac{M'(t)}{M(t)}\,,$$

and integrating the resulting equation over $[t_1, t_2]$ we obtain from (2.15), (2.17), and (2.18) that

$$\int_{t_1}^{t_2} \left(\|Au(s)\|^2 + \frac{\rho}{2} \left(\frac{M'(s)}{M(s)} \right)^2 \right) ds$$
$$\leq (\rho H(0))^{\frac{1}{2}} \int_t^{t+1} \frac{\|A^{1/2}u'\|}{M(s)^{\frac{1}{2}}} \, ds + \rho \sum_{j=1}^{2} \frac{\|A^{1/2}u'(t_j)\|}{M(t_j)^{\frac{1}{2}}} + \int_t^{t+1} \frac{\|A^{1/2}u'\|}{M(s)^{\frac{1}{2}}} \, ds$$
$$\leq \left(H(0)^{\frac{1}{2}} + 5 \right) D(t) . \tag{2.19}$$

Moreover, we observe from (2.15) and (2.19) that

$$\int_{t_1}^{t_2} H(s) \, ds \leq (\rho H(0))^{\frac{1}{2}} \int_t^{t+1} \frac{\|A^{1/2}u'(s)\|}{M(s)^{\frac{1}{2}}} \, ds + \int_{t_1}^{t_2} \|Au(s)\|^2 \, ds$$
$$\leq \left(2H(0)^{\frac{1}{2}} + 5 \right) D(t) . \tag{2.20}$$

Integrating (2.13) over $[t, t_2]$, we have from (2.4) and (2.5) that

$$H(t) = H(t_2) + 2 \int_{t_1}^{t_2} \left(a(s) + \frac{\rho}{2} \frac{M'(s)}{M(s)} \right) \frac{\|A^{1/2}u'(s)\|^2}{M(s)} \, ds$$
$$\leq 2 \int_{t_1}^{t_2} H(s) \, ds + 3 \int_t^{t+1} a(s) \frac{\|A^{1/2}u'(s)\|^2}{M(s)} \, ds$$

and from (2.20) and (2.16) that

$$H(t) \leq \left(2H(t)^{\frac{1}{2}} + 5 \right) D(t) + 3a(t+1)D(t)^2$$
$$\leq 5 \left(H(0)^{\frac{1}{2}} + 1 \right) D(t) ,$$

where we used the facts $a(t+1)D(t)^2 \leq H(t) \leq H(0)$ (by (2.16)) and $a(t+1) \leq 1$ at the last inequality, and hence, we have from (2.16) that

$$H(t)^2 \leq 5^2 \left(H(0)^{\frac{1}{2}} + 1 \right)^2 D(t)^2$$
$$\leq 10^2 (H(0) + 1)(2 + t)^p (H(t) - H(t + 1))$$
$$\leq 2^{p+7} (H(0) + 1)(1 + t)^p (H(t) - H(t + 1)) .$$

Then we observe

$$H(t + 1)^{-1} - H(t)^{-1} = \int_0^1 \frac{d}{d\theta} \left(\theta H(t + 1) + (1 - \theta)H(t) \right)^{-1} \, d\theta$$
$$\geq H(t)^{-2} (H(t) - H(t + 1))$$
$$\geq k^{-1} (1 + t)^{-p} , \qquad k = 2^{p+7} (H(0) + 1) ,$$

and we have

$$H(t+1)^{-1} \geq \inf_{0\leq s<1} H(s)^{-1} + k^{-1}\int_0^t (2+t-x)^{-p}\,dx \quad \text{for} \quad t\geq 0$$

or

$$H(t)^{-1} \geq H(0)^{-1} + k^{-1}(1-p)^{-1}\left((1+t)^{1-p} - 2^{1-p}\right) \quad \text{for} \quad t\geq 1$$

which gives the desired estimate (2.12). □

3 Main result

Theorem 3.1 *Let the initial data* (u_0, u_1) *belong to* $\mathcal{D}(A)\times\mathcal{D}(A^{1/2})$ *and* $u_0 \neq 0$ *and* $p \leq 1/3$. *Suppose that the coefficient* ρ *or the initial data* (u_0, u_1) *are small such that*

$$\rho H(0) < K \tag{3.1}$$

where K *is a positive constant given by*

$$K \equiv \inf_{t\geq 0} \frac{1}{4^2(1+t)^{2p}}\left(1 + \frac{H(0)}{k(1-p)}\left[(1+t)^{1-p} - 2^{1-p}\right]^+\right). \tag{3.2}$$

Then, the problem (1.1) *admits a unique global solution* $u(t)$ *in the class* $C^0([0,\infty);\mathcal{D}(A))\cap C^1([0,\infty);\mathcal{D}(A^{1/2}))\cap C^2([0,\infty);L^2)$, *and this solution* $u(t)$ *satisfies that*

$$C'(1+t)^{-(1+p)} \leq \|A^{1/2}u(t)\|^2 \leq C(1+t)^{-(1-p)} \tag{3.3}$$

$$C'(1+t)^{-(1+p)} \leq \|Au(t)\|^2 \leq C(1+t)^{-(1-p)} \tag{3.4}$$

$$\|A^{1/2}u'(t)\|^2 \leq C(1+t)^{-2(1-p)} \quad \textit{for} \quad t\geq 0 \tag{3.5}$$

where C *and* C' *are some positive constants.*

Proof. Let $u(t)$ be a solution of (1.1) on $[0,T]$. Since $M(0) > 0$ (by $u_0 \neq 0$), putting

$$T_1 \equiv \sup\left\{\, t\in[0,\infty) \;\middle|\; M(s) > 0 \quad \text{for} \quad 0\leq s<t \,\right\},$$

we see that $T_1 > 0$. If $T_1 < T$, then

$$M(t) > 0 \quad \text{for} \quad 0\leq t<T_1 \quad \text{and} \quad M(T_1) = 0\,. \tag{3.6}$$

Since $4^2\rho H(0) < 1$ (by (3.1)), putting

$$T_2 \equiv \sup\left\{t\in[0,\infty) \;\middle|\; 4^2\rho(1+s)^{2p}H(s) < 1 \text{ for } 0\leq s<t\right\},$$

we see that $T_2 > 0$. If $T_2 < T_1$, then

$$4^2 \rho (1+t)^{2p} H(t) < 1 \quad \text{for} \quad 0 \le t < T_2 \quad \text{and} \quad 4^2 \rho (1+T_2)^{2p} H(T_2) = 1 \quad (3.7)$$

or

$$4(\rho H(t))^{\frac{1}{2}} < a(t) \quad \text{for} \quad 0 \le t \le T_2 \,. \qquad (3.8)$$

Thus, from (3.1) and (3.8) we observe that if $p \le 1/3$,

$$\rho H(0) < K \le \frac{1}{4^2(1+t)^{2p}} \left(1 + \frac{H(0)}{k(1-p)} \left[(1+t)^{1-p} - 2^{1-p} \right]^+ \right) ,$$

that is,

$$4^2 \rho (1+t)^{2p} \left(H(0)^{-1} + k^{-1}(1-p)^{-1} \left[(1+t)^{1-p} - 2^{1-p} \right]^+ \right)^{-1} < 1 \,,$$

and then, from Proposition 2.5,

$$4^2 \rho (1+t)^{2p} H(t) < 1 \quad \text{for} \quad 0 \le t \le T_2$$

which is a contradiction to (3.7), and hence, we have that $T_2 \ge T_1$.

Moreover, from Proposition 2.4 and (3.8) we observe

$$M(t) \ge C'(1+t)^{-(1+p)} > 0 \quad \text{for} \quad 0 \le t \le T_1$$

which is a contradiction to (3.6), and hence, we have that $T_1 \ge T$.

Thus, we conclude that $M(t) > 0$ and $\|A^{1/2}u'(t)\| + \|Au(t)\| \le C$ for $0 \le t \le T$. Therefore, the local solution $u(t)$ of (1.1) in the sense of Proposition 2.1 can be continued globally in time. Also, from Propositions 2.3, 2.4, and 2.5 we obtain the desired estimates (3.3), (3.4), and (3.5). □

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