Almost Contact Metric Submersions in Kenmotsu Geometry by

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Abstract

In this paper, we discuss some geometric properties of Riemannian submersions whose total space is a manifold through various classes of Kenmotsu structures. The study focuses on the superminimality of the fibres. This property facilitates the transference of the structure from the base to the total space.

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Introduction

Almost contact metric submersions are Riemannian submersions whose total space is furnished with an almost contact metric structure (φ, ξ, η, g). This theory, initiated by D. Chinea [2], and B. Watson [14] independentely each other, continue to fascinate a number of differential geometers; For instance, we can cite [4], [8] and [9] among many others.

From D. Chinea and C. Gonzalez [5], it is known that there are 4.096 classes of almost contact structures. These manifolds are grouped in Sasakian, cosymplectic and Kenmotsu types. In [10], K.Kenmotsu introduced a class of almost contact Riemannian manifolds which are neither cosymplectic nor Sasakian.

In this paper, we would like to understand what happens in the Kenmotsu type by the use of Riemannian submersions. One of the main problems in the study of almost contact metric submersions is the relationships between the properties of the total space, the base space and the fibres.

The present paper is organized as follows. Section 1 is devoted to the preliminaries on almost Hermitian and almost contact metric manifolds. Here, various classes of manifolds in Kenmotsu type and the diagram of their inclusions are presented. Section 2 deals with almost contact metric submersions. In this section, we examine the structure of the fibres, the transference of the structure from the total to the base space or to the fibres. We show that the superminimality of the fibres is a tool in the transference of the structure from the base to the total space. In this case, we have found that the criterion of Watson, [15], for this transference does not apply to some other classes such as G_2-*Kenmotsu* and *almost trans-Kenmotsu.*

1 Preliminaries on Manifolds

1.1 Almost Hermitian Manifolds

By an almost Hermitian manifold, one understands a Riemannian manifold, (M, g), of dimension $2m$, furnished with a tensor field , J, of type $(1, 1)$ satisfying the following two conditions:

(i) $J^2D = -D$, and

(ii) $g(JD, JE) = g(D, E)$, for all $D, E \in \chi(M)$.

Any almost Hermitian manifold admits a differential $2-$from, Ω, called the fundamental form or the Kähler form, defined by

$$\Omega(D, E) = g(D, JE).$$

The codifferential of Ω is given by

$$\delta\Omega(D) = -\sum_{i=1}^{m}\{(\nabla_{E_i}\Omega)(E_i, D) + (\nabla_{JE_i}\Omega)(JE_i, D)\}$$

The Lee $1-$form is defined by $\theta(D) = \frac{1}{m-1}\delta\Omega(JD)$.

Almost Hermitian structures have been completely classified by A. Gray and L.M. Hervella [7]. We just recall the defining relations of some classes which will be used in this study.

An almost Hermitian manifold (M^{2m}, g, J) is said to be :

(a) *Kählerian* if $d\Omega(D, E, G) = 0$ and $N_J = 0$, where N_J denotes the Nijenhuis tensor of J;

(b) *almost Kählerian* (or W_2-manifold) if $d\Omega(D, E, G) = 0$;

(c) *nearly Kählerian* (or W_1-manifold) if $(\nabla_D\Omega)(D, E) = 0$;

(d) W_3-*manifold* if $(\nabla_D\Omega)(E, G) - (\nabla_{JD}\Omega)(JE, G) = 0 = \delta\Omega$;

(e) *semi-Kählerian* (or $W_1 \oplus W_2 \oplus W_3$-manifold) if $\delta\Omega = 0$;

(f) $W_1 \oplus W_3$-*manifold* if $(\nabla_D\Omega)(D,E) - (\nabla_{JD}\Omega)(JD,E) = 0 = \delta\Omega$;

(g) G_1-*manifold* if $(\nabla_D\Omega)(D,E) - (\nabla_{JD}\Omega)(JD,E) = 0$;

(h) *Hermitian* or ($W_3 \oplus W_4$-manifold) if $N_J = 0$ or equivalently $(\nabla_D\Omega)(E,G) - (\nabla_{JD}\Omega)(JE,G) = 0$.

(i) a G_2-*manifold* or ($W_2 \oplus W_3 \oplus W_4$-manifold) if $\mathcal{G}\{(\nabla_D\Omega)(E,G) - (\nabla_{JD}\Omega)(JE,G)\} = 0$ or $\mathcal{G}\{g(N_J(D,E),JG)\} = 0$.

(j) *quasi Kählerian* or ($W_1 \oplus W_2$-manifold) if $(\nabla_D\Omega)(E,G) + (\nabla_{JD}\Omega)(JE,G) = 0$.

(k) $W_2 \oplus W_3$-*manifold* if $\mathcal{G}\{(\nabla_D\Omega)(D,E) - (\nabla_{JD}\Omega)(JD,E)\} = 0 = \delta\Omega$.

(l) *locally conformal almost Kähler* ($W_2 \oplus W_4$-manifold) if $d\Omega = \Omega\Lambda\theta$ or $\mathcal{G}\left\{(\nabla_D\Omega)(E,G) - \frac{1}{m-1}\Omega(D,E)\,\delta\Omega(JG)\right\} = 0$,

where $\mathcal{G}$ denotes the cyclic sum over D, E and G.

1.2 Various Classes of Kenmotsu Manifolds

Since Kenmotsu manifolds constitute a fragment of contact geometry, we begin by recalling some background notions in the latter.

An almost contact structure on a differentiable manifold, M, is a triple (φ, ξ, η) where:

(i) ξ is a characteristic vector field,

(ii) η is a differential $1-$form such that $\eta(\xi) = 1$, and

(iii) φ is a tensor field of type $(1,1)$ satisfying

$$\varphi^2 D = -D + \eta(D)\xi,$$

for all $D \in \chi(M)$.

As in the case of almost Hermitian manifolds, the fundamental $2-$form, ϕ, of an almost contact metric manifold is defined by

$$\phi(D,E) = g(D,\varphi E).$$

Among some remarkable identities we have:

(1) $(\nabla_D\eta)E = g(E,\nabla_D\xi)$;

(2) $2d\eta(D,E) = (\nabla_D\eta)E - (\nabla_E\eta)D$.

$$3d\phi(D,E,G) = \mathcal{G}\left\{(\nabla_D\phi)(E,G)\right\}.$$

Let $\{E_1, ..., E_m, \varphi E_1, ..., \varphi E_m, \xi\}$ be a local $\varphi-$basis of an open subset of M, then the codifferential δ is given by

$$\delta\phi(D) = -\sum_{i=1}^{m}\left\{(\nabla_{E_i}\phi)(E_i, D) + (\nabla_{\varphi E_i}\phi)(\varphi E_i, D)\right\} - (\nabla_\xi\phi)(\xi, D);$$

$$\delta\eta = -\sum_{i=1}^{m}\left\{(\nabla_{E_i}\eta)E_i + (\nabla_{\varphi E_i}\eta)\varphi E_i\right\}.$$

Almost contact metric manifolds are extensively studied in [1].

The analogous of the Lee form is the $1-$form, ω, defined by

$$\omega(D) = \frac{1}{m}(\delta\phi(\varphi D) - \eta(D)\delta\eta).$$

Let us recall the well known structures in the topic of Kenmotu manifolds. An almost contact metric manifold is said to be:

(1) *almost Kenmotsu* if $d\phi(D,E,G) = \frac{2}{3}\mathcal{G}\left\{\eta(D)\phi(E,G)\right\};$

(2) *Kenmotsu* if $d\phi(D,E,G) = \frac{2}{3}\mathcal{G}\left\{\eta(D)\phi(E,G)\right\}$, $d\eta = 0$ and $N^{(1)} = 0$; where $N^{(1)} = N_\varphi + 2d\eta \otimes \xi$ with N_φ the Nijenhuis tensor of φ.

(3) G_1-*Kenmotsu* if $(\nabla_D\phi)(D,E) - (\nabla_{\varphi D}\phi)(\varphi D,E) - \eta(D)\phi(E,D) = 0 = d\eta;$

(4) G_1-*semi-Kenmotsu* if it is G_1-Kenmotsu and $\delta\phi = 0$;

(5) G_2-*Kenmotsu* if $\mathcal{G}\left\{(\nabla_D\phi)(E,G) - (\nabla_{\varphi D}\phi)(\varphi E,G) - \eta(D)\phi(E,G)\right\} = 0 = d\eta;$

(6) G_2-*semi-Kenmotsu* if it is G_2-Kenmotsu and $\delta\phi = 0$;

(7) *nearly Kenmotsu* if $(\nabla_D\varphi)D = -\eta(D)\varphi D$ and $d\eta = 0$;

(8) *semi-Kenmotsu normal* if $(\nabla_D\phi)(E,G) - (\nabla_{\varphi D}\phi)(\varphi E,G) = \eta(E)\phi(G,D)$, $\delta\phi = 0$ and $d\eta = 0$;

(9) *quasi-Kenmotsu* if $(\nabla_D\phi)(E,G) + (\nabla_{\varphi D}\phi)(\varphi E,G) = \eta(E)\phi(G,D) + 2\eta(G)\phi(D,E)$ and $d\eta = 0$;

(10) *almost trans-Kenmotsu* if $\mathcal{G}\left\{(\nabla_D\phi)(E,G) - \frac{1}{m}\phi(D,E)\delta\phi(\varphi G) - 2\eta(D)\phi(E,G)\right\} = 0$ and $d\eta = 0$;

(11) *generalized Kenmotsu* if $(\nabla_D\phi)(E,G) - (\nabla_{\varphi D}\phi)(\varphi E,G) = \eta(E)\phi(G,D)$ and $d\eta = 0$.

The latter class is introduced in [13].

These various classes are related as shown in the following diagram.

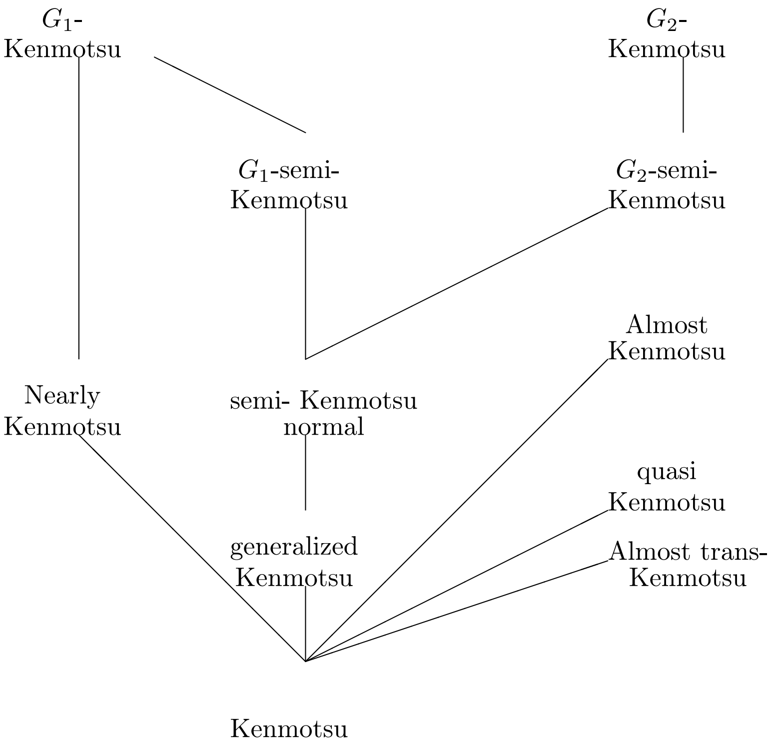

Figure 1. Diagram of the strict inclusions.

2 Almost Contact Metric Submersions

In [11], O'Neill has defined a Riemannian submersion as a surjective mapping

$$\pi : M \longrightarrow B$$

between two Riemannian manifolds such that

(i) π is of maximal rank;

(ii) $\pi_{*|(Ker\pi_*)^\perp}$ is a linear isometry.

The tangent bundle $T(M)$, of the total space M, admits an orthogonal decomposition

$$T(M) = V(M) \oplus H(M),$$

where $V(M)$ and $H(M)$ denote respectively the vertical and horizontal distributions. We denote by $\mathcal{V}$ and $\mathcal{H}$ the vertical and horizontal projections respectively. A vector field X of the horizontal distribution is called a basic vector field if it is $\pi-$related to a vector field X_* of the base space B. Such a vector field means that $X_* = \pi_* X$.

On the base space, tensors and other objects will be denoted by a prime ' while those tangent to the fibres will be specified by a carret ˆ. Herein, vector fields tangent to the fibres will be denoted by U, V and W.

Let $(M^{2m+1}, g, \varphi, \xi, \eta)$ and $(M'^{2m'+1}, g', \varphi', \xi', \eta')$ be two almost contact metric manifolds. By an almost contact metric submersion of type I, in the sense of Watson [14], one understands a Riemannian submersion

$$\pi : M^{2m+1} \to M'^{2m'+1}$$

satisfying

(i) $\pi_*\varphi = \varphi'\pi_*$,

(ii) $\pi_*\xi = \xi'$.

When the base space is an almost Hermitian manifold, $(M'^{2m'}, g', J')$, the Riemannian submersion

$$\pi : M^{2m+1} \to M'^{2m'}$$

is called an almost contact metric submersion of type II, if $\pi_*\varphi = J'\pi_*$, [14].

Various classes of Riemannian submersions are presented in [6].

2.1 Fundamental Properties

Now, we overview some of the fundamental properties of these submersions.

Proposition 2.1. *Let $\pi : M^{2m+1} \longrightarrow M'^{2m'+1}$ be an almost contact metric submersion of type I. Then*

(a) $\pi^*\phi' = \phi$;

(b) $\pi^*\eta' = \eta$;

(c) $\eta(U) = 0$ *for all* $U \in V(M)$;

(d) $\mathcal{H}(\nabla_X \varphi)Y$ *is the basic vector field associated to* $(\nabla'_{X_*}\varphi')Y_*$ *if* X *and* Y *are basic.*

Proof. See Watson [14]. □

Proposition 2.2. *Let* $\pi : M^{2m+1} \longrightarrow M'^{2m'}$ *be an almost contact metric submersion of type II. Then*

(a) $\pi^*\Omega' = \phi$;

(b) $\eta(X) = 0$ *for all* $X \in H(M)$;

(c) $\mathcal{H}(\nabla_X \varphi)Y$ *is the basic vector field associated to* $(\nabla'_{X_*}J')Y_*$ *if* X *and* Y *are basic.*

Proof. See again Watson [14]. □

2.2 Transference of Structure

Proposition 2.3. *Let* $\pi : M^{2m+1} \longrightarrow M'^{2m'+1}$ *be an almost contact metric submersion of type I. If the total space is generalized Kenmotsu, Kenmotsu, almost Kenmotsu, quasi Kenmotsu, nearly Kenmotsu or* G_i-*Kenmotsu for* $i \in \{1, 2\}$ *, then the fibres are respectively Hermitian, Kähler, almost Kähler, quasi Kähler, nearly Kähler or* G_i-*manifolds for* $i \in \{1, 2\}$ *.*

Proof. Let M^{2m+1} be a generalized Kenmotsu manifold. Considering three vector fields U, V and W tangent to the fibres, we have

$$(\nabla_U \phi)(V, W) - (\nabla_{\varphi U} \phi)(\varphi V, W) = \eta(V)\phi(W, U)$$

and $d\eta = 0$; The vanishing of η on vertical vector fields gives to

$$(\nabla_U \phi)(V, W) - (\nabla_{\varphi U} \phi)(\varphi V, W) = 0$$

which corresponds to

$$(\nabla_U \Omega)(V, W) - (\nabla_{JU} \Omega)(JV, W) = 0$$

which is the defining relation of a Hermitian manifold. □

Recall that, the O'Neill tensors of configuration T and A are defined, on the total space of a Riemannian submersion by setting

$$T_D E = \mathcal{H}\nabla_{\mathcal{V}D}\mathcal{V}E + \mathcal{V}\nabla_{\mathcal{V}D}\mathcal{H}E,$$

$$A_D E = \mathcal{V}\nabla_{\mathcal{H}D}\mathcal{H}E + \mathcal{H}\nabla_{\mathcal{H}D}\mathcal{V}E.$$

These tensors play an important role in the study of the geometry of the fibres and the horizontal distribution respectively.

Using the latest, which is the integrability tensor, D. Chinea [3] has defined an associated tensor A^* on horizontal vector fields by putting

$$A^*(X,Y) = A_X\varphi Y - A_{\varphi X}Y$$

and has established the following structure equations

$$\delta\phi(U) = \delta\hat{\phi}(U) + \frac{1}{2}g(trA^*, U); \quad (2.1)$$

$$\delta\phi(X) = \delta\phi'(X_*) + g(H, \varphi X); \quad (2.2)$$

$$\delta\eta = \delta\eta' \circ \pi - g(H, \xi), \quad (2.3)$$

where H denotes the mean curvature vector field of the fibres and trA^* is the trace of A^*.

Concerning the Lee forms ω and θ, the above equations lead to the following

Lemma 2.1. *Let $\pi : M^{2m+1} \longrightarrow M'^{2m'+1}$ be an almost contact metric submersion of type I, then $\theta(U) = \hat{\omega}(U)$ if and only if $trA^* = 0$.*

Proof. The vanishing of η on the vertical distribution leads to

$$\hat{\omega}(U) = \frac{1}{m-1}\delta\hat{\phi}(\hat{\varphi}U).$$

But, from equation (2.1), we have $\delta\hat{\phi}(\hat{\varphi}U) = \delta\phi(\varphi U)$ if and only if $trA^* = 0$. Since the fibres of an almost contact metric submersion of type I are almost Hermitian manifolds, we have $\hat{\varphi}U = JU$ and $\delta\hat{\phi} = \delta\Omega$ so that $\hat{\omega}(U) = \theta(U)$ as required. □

Proposition 2.4. *If the total space of an almost contact metric submersion of type I is semi-Kenmotsu normal or G_i-semi-Kenmotsu for $i \in \{1,2\}$, then the fibres are W_3-manifolds or $W_i \oplus W_3-$manifolds respectively if and only if $trA^* = 0$.*

Proof. Let us consider the case of semi-Kenmotsu normal. On vertical vector fields U, V and W tangent to the fibres, we have
$(\nabla_U\phi)(V,W) - (\nabla_{\varphi U}\phi)(\varphi V, W) = \eta(V)\phi(W,U)$, $\delta\phi = 0$ and $d\eta = 0$;
the vanishing of η on vertical vector fields leads to
$(\nabla_U\phi)(V,W) - (\nabla_{\varphi U}\phi)(\varphi V, W) = 0$, $\delta\phi = 0$ and $d\eta = 0$;

In the light of equation (2.1), $\delta\hat{\phi} = 0$ if and only if $trA^* = 0$. On the other hand, since $\hat{\varphi} = J$ on the fibres of a type I submersion, we obtain $(\nabla_U\Omega)(V,W) - (\nabla_{JU}\Omega)(JV,W) = 0 = \delta\Omega$. Other cases are treated in the same way. □

Proposition 2.5. *Let $\pi : M^{2m+1} \longrightarrow M'^{2m'+1}$ be an almost contact metric submersion of type I. If the total space is almost trans-Kenmotsu , then the fibres are $W_2 \oplus W_4-$manifolds if and only if $trA^* = 0$.*

Proof. Analogous to the preceding. □

Let us look to the analogous properties of submersions of type II.

Proposition 2.6. *Let $\pi : M^{2m+1} \longrightarrow M'^{2m'}$ be an almost contact metric submersion of type II. If the total space is G_i-semi-Kenmotsu for $i \in \{1, 2\}$, almost trans-Kenmotsu or semi-Kenmotsu normal, then the fibres inherit the structure of the total space if and only if $trA^* = 0$.*

Proof. Note that all these manifolds are defined by the use of codifferential, $\delta\phi$, of the fundamental $2-$form ϕ. Using equation (2.1), we have that $\delta\hat{\phi} = \delta\phi$ if and only if $trA^* = 0$ on vertical vector fields. □

Proposition 2.7. *Let $\pi : M^{2m+1} \longrightarrow M'^{2m'}$ be an almost contact metric submersion of type II. If the total space is an almost trans-Kenmotsu manifold, then the base space is a $W_2 \oplus W_4-$manifold if and only if the fibres are minimal.*

Proof. Because of the vanishing of η on horizontal vector fields, we have

$$\mathcal{G}\left\{(\nabla_X\phi)(Y,Z) - \frac{1}{m}\phi(X,Y)\delta\phi(\varphi Z)\right\} = 0.$$

Taking into account that the base space is an almost Hermitian manifold, we get

$$\mathcal{G}\left\{(\nabla'_{X_*}\Omega')(Y_*,Z_*) - \frac{1}{m-1}\Omega'(X_*,Y_*)\delta\Omega'(J'Z_*)\right\} = 0,$$

which is the defining relation of a locally conformal almost Kähler (or a $W_2 \oplus W_4-$manifold). Recall that, with equation (2.2) in mind, $\delta\phi = \delta\Omega'$ if and only if $H = 0$. □

Proposition 2.8. *Let $\pi : M^{2m+1} \longrightarrow M'$ be an almost contact metric submersion of type I or type II. If the total space is endowed with the nearly Kenmotsu structure, then,*

(a) $T_U\varphi U = \varphi T_U U$,

(b) $T_\xi\xi = 0$,

(c) $A_X\varphi X = 0$,

(d) $A_\xi\xi = 0$.

Proof. Remember that a nearly Kenmotsu manifold is defined by

$$(\nabla_D\varphi)D = -\eta(D)\varphi D.$$

In the case of a submersion of type I, the vanishing of η on vertical vector fields gives $\eta(U) = 0$ so that the defining relation becomes $(\nabla_U\varphi)U = 0$ from which the horizontal projection gives

$$T_U\varphi U = \varphi T_U U.$$

Considering the case of a submersion of type II, the horizontal projection gives also $(\nabla_U \varphi)U = 0$ because φU is vertical. We then get the proof of (a).

Now, let us examine assertion (b). If we have a type I submersion, ξ is horizontal (basic). In this case, $T_\xi \xi = 0$ according to the fact that $T_E = T_{\mathcal{V}E}$. If we have a submersion of type II, one has $(\nabla_\xi \varphi)\xi = 0$ from which $\nabla_\xi \xi = 0$ follows. Since ξ is vertical, the horizontal projection of $\nabla_\xi \xi = 0$ gives $T_\xi \xi = 0$.

Concerning assertion (c), it is clear that the relation becomes

$$(\nabla_X \varphi)X = -\eta(X)\varphi X.$$

If we have a submersion of type I, since φX is horizontal, the vertical projection gives $\mathcal{V}(\nabla_X \varphi)X = 0$ from which $A_X \varphi X = \varphi A_X X$ follows. It can be shown that $A_X \varphi X = 0$.

If we have a submersion of type II, the relation gives $(\nabla_X \varphi)X = 0$ because $\eta(X) = 0$ and thus, $A_X \varphi X = 0$ is deduced.

Considering assertion (d). In the case of a submersion of type I, we have $(\nabla_\xi \varphi)\xi = 0$ as above from which $\nabla_\xi \xi = 0$ follows. Since ξ is horizontal, the vertical projection of $\nabla_\xi \xi = 0$ gives $A_\xi \xi = 0$. If we have a submersion of type II, it is known that ξ is vertical from which the proof follows. □

Proposition 2.9. *Let $\pi : M^{2m+1} \longrightarrow M'$ be an almost contact metric submersion of type I or type II. If the total space is nearly Kenmotsu , then the fibres are minimal.*

Proof. Since, by Proposition 2.8, $T_U \varphi U = \varphi T_U U$, it is easy to show that the fibres are minimal. □

Some implications of the minimality of the fibres

Proposition 2.10. *Let $\pi : M^{2m+1} \longrightarrow M'^{2m'+1}$ be an almost contact metric submersion of type I. If the total space is semi-Kenmotsu normal, G_i-semi-Kenmotsu for $i \in \{1, 2\}$, or almost trans-Kenmotsu, then the base space inherits the structure of the total space if and only if the fibres are minimal.*

Proof. The manifolds under consideration beeing defined with the codifferential, it is clear that, with the use of equation (2.2), $\delta\phi' = \delta\phi$ if and only if $H = 0$. □

Proposition 2.11. *Let $\pi : M^{2m+1} \longrightarrow M'^{2m'}$ be an almost contact metric submersion of type II. If the total space is G_i-semi-Kenmotsu for $i \in \{1, 2\}$ or semi-Kenmotsu normal, then the base space is respectively $W_i \oplus W_3$ or W_3-manifold if and only if the fibres are minimal.*

Proof. Let us consider the case of G_1-semi-Kenmotsu which means that $i = 1$. On basic vector fields X and Y, we have

$$(\nabla_X \phi)(X, Y) - (\nabla_{\varphi X} \phi)(\varphi X, Y) = 0$$

because of the vanishing of η on horizontal vector fields.

With the fact that $\pi^*\Omega' = \phi$, the above relation becomes

$$(\nabla'_{X_*}\Omega')(X_*, Y_*) - (\nabla'_{J'X_*}\Omega')(J'X_*, Y_*) = 0.$$

On the other hand, since the total space is defined with $\delta\phi = 0$, we can use the Chinea equation (2.2) which leads to $\delta\Omega' = 0$ if and only if $H = 0$. □

2.3 Superminimality of the fibres

Now we want to examine the superminimality of the fibres. Let $(M^{2m+1}, g, \varphi, \xi, \eta)$ be an almost contact metric manifold and $\bar{M}$ a $\varphi-$invariant submanifold of M. If, $\nabla_V\varphi = 0$ for all V tangent to $\bar{M}$, then $\bar{M}$ is said to be superminimal.

In order to verify the superminimality of the fibres of an almost contact metric submersion of type I, there are four components of $g(\nabla_V\varphi)D, E)$ to be considered on the total space M. From [12] we recall that

SM-1) $g((\nabla_V\varphi)U, W) = g(\hat{\nabla}_V(\hat{J}U) - \hat{J}\hat{\nabla}_V U, W)$,

SM-2) $g((\nabla_V\varphi)U, X) = g(T_V(\varphi U) - \varphi(T_V U), X)$,

SM-3) $g((\nabla_V\varphi)X, U) = -g((\nabla_V\varphi)U, X)$,

SM-4) $g((\nabla_V\varphi)X, Y) = -g(A_{\varphi X}Y + A_X(\varphi Y), V)$.

In the case of an almost contact metric submersion of type II, we easily find

SM-5) $g((\nabla_V\varphi)U, W) = g(\hat{\nabla}_V(\hat{\varphi}U) - \hat{\varphi}\hat{\nabla}_V U, W)$,

SM-6) $g((\nabla_V\varphi)U, X) = g(T_V(\varphi U) - \varphi(T_V U), X)$,

SM-7) $g((\nabla_V\varphi)X, U) = -g((\nabla_V\varphi)U, X)$,

SM-8) $g((\nabla_V\varphi)X, Y) = -g(A_{\varphi X}Y + A_X(\varphi Y), V)$.

It is clear that $SM - 1)$ implies that if the fibres are superminimal, then they are Kähler.

Proposition 2.12. *Let $\pi : M^{2m+1} \longrightarrow M'^{2m'+1}$ be an almost contact metric submersion of type I. If the total space is a Kenmotsu manifold, then the fibres cannot be superminimal.*

Proof. Suppose that the fibres are superminimal. This means that $\nabla_U\varphi = 0$ for all vector fields U tangent to the fibres. But on Kenmotsu manifold we have

$$\begin{aligned} 0 &= g((\nabla_U\varphi)\varphi U, \xi) \\ &= g(\varphi U, \varphi U)g(\xi, \xi) \\ &= \|U\|^2. \end{aligned}$$

If $\|U\|^2 = 0$ then $U = 0$ which is not true. Thus, the fibres cannot be superminimal. □

We can say something concerning the integrability of the horizontal distribution.

Proposition 2.13. *Let $\pi : M^{2m+1} \longrightarrow M'^{2m'}$ be an almost contact metric submersion of type II with the total space a nearly Kenmotsu manifold. If the fibres are superminimal, then the horizontal distribution is completely integrable.*

Proof. Since η vanishes on horizontal vector fields, the defining relation of a nearly Kenmotsu manifold gives

$$(\nabla_X \varphi)X = 0.$$

On an almost contact metric submersion of type II with M, nearly Kenmotsu, $(\nabla_X \varphi)X = -\eta(X)\varphi X$ for any horizontal vector field X. Therefore, $A_X \varphi X = 0$. The usual polarization trick implies that $A_X \varphi Y = \varphi A_X Y$. Combining this with expression $SM - 8)$ yields A$\equiv 0$. □

Proposition 2.14. *Let $\pi : M^{2m+1} \longrightarrow M'^{2m'+1}$ be an almost contact metric submersion of type I with the total space a G_1-Kenmotsu manifold. If the fibres are superminimal, then the horizontal distribution is completely integrable.*

Proof. Let X be a horizontal vector fields and U a vertical one. According to the defining relation of a G_1-Kenmotsu structure, we have

$$(\nabla_X \phi)(X, U) - (\nabla_{\varphi X} \phi)(\varphi X, U) = \eta(X)\phi(X, U).$$

Thus we obtain

$$-g(A_X X, \varphi U) - g(A_X \varphi X, U) - g(A_{\varphi X} \varphi X, \varphi U) + g(A_{\varphi X} X, U) = 0,$$

yielding $2g(A_{\varphi X} X, U) = 0$, from which one gets $A_{\varphi X} X = 0$.
Combining this result with the fact that expression $SM - 4)$ vanishes, we have A$\equiv 0$. □

In the following, we will show that the superminimality of the fibres is a tool in the transference of the structure from the base to the total space.

Lemma 2.2. *Let $\pi : M^{2m+1} \longrightarrow M'^{2m'+1}$ be an almost contact metric submersion of type I. Suppose that $d\eta' = 0$ on the base space. If the fibres are superminimal, then $d\eta = 0$ on the total space.*

Proof. In order to see that $d\eta = 0$, we begin by assuming that X and Y are basic vector fields on the total space. Then $d\eta(X, Y) = d\eta'(X_*, Y_*) = 0$. The vanishing of expression $SM - 2)$ implies, along with $A_{\varphi X} = 0$ that A$\equiv 0$. Now

$$\begin{aligned} 2d\eta(X, U) &= (\nabla_X \eta)U - (\nabla_U \eta)X \\ &= g(X, \nabla_U \xi) - g(U, \nabla_X \xi) \\ &= g(X, \nabla_U \xi) - g(U, A_X \xi) \\ &= g(X, \nabla_U \xi). \end{aligned}$$

The superminimality of the fibres implies that

$$\begin{aligned} 0 &= g((\nabla_U \varphi)\xi, X) \\ &= g(\nabla_U \varphi\xi, X) - g(\varphi\nabla_U \xi, X) \\ &= g(\nabla_U \xi, \varphi X). \end{aligned}$$

Thus, $\nabla_U \xi$ is $g-$orthogonal to all vector fields except, perhaps, ξ. Recall that $\|\xi\|^2 = g(\xi, \xi)$ is constant 1, so that $g(\nabla_U \xi, \xi) = 0$.
Hence $d\eta(X, U) = 0$ and $d\eta(U, X) = 0$.

Recall, too, that the Lie bracket $[U, V]$ is vertical from the complete integrability of the vertical distribution.

Then

$$d\eta(U, V) = \frac{1}{2}\{U\eta(V) - V\eta(U) - \eta([U, V])\} = 0$$

because η vanishes on the vertical distribution. □

The above Lemma 2.2 applies to nearly Kenmotsu, quasi Kenmotsu and generalized Kenmotsu manifolds among many others.

Proposition 2.15. *Let $\pi : M^{2m+1} \longrightarrow M'^{2m'+1}$ be an almost contact metric submersion of type I. Assume that the base space is nearly Kenmotsu, a G_1-Kenmotsu or G_1-semi-Kenmotsu. If the fibres are superminimal, then the total space is nearly Kenmotsu, G_1-Kenmotsu or G_1-semi-Kenmotsu respectively.*

Proof. Let us consider the case of the nearly Kenmotsu structure. Lemma 2.2 implies that $d\eta = 0$ on the total space. Since η vanishes on the vertical distribution, we need only to show that $(\nabla_U \varphi)U = 0$ and that

$$0 = (\nabla_X \varphi)X + \eta(X)\varphi X.$$

Let X be basic, then

$$(\nabla_X \varphi)X + \eta(X)\varphi X = (\nabla'_{X_*} \varphi')X_* + \eta'(X_*)\varphi' X_* = 0.$$

Clearly, $(\nabla_U \varphi)U = 0$ because the fibres are superminimal. Therefore, the total space is nearly Kenmotsu.

Now, let us consider the case of G_1-Kenmotsu structure. In the preceding Lemma 2.2 it is shown that if $d\eta' = 0$ on the base space then the total space also verifies $d\eta = 0$.

There are four components which must vanish to verify the
G_1-Kenmotsu structure on the total space. We have

$G_1 - K-1)$ $(\nabla_U \phi)(U, V) - (\nabla_{\varphi U} \phi)(\varphi U, V) - \eta(U)\phi(V, U)$;
$G_1 - K-2)$ $(\nabla_U \phi)(U, X) - (\nabla_{\varphi U} \phi)(\varphi U, X) - \eta(U)\phi(X, U)$;
$G_1 - K-3)$ $(\nabla_X \phi)(X, V) - (\nabla_{\varphi X} \phi)(\varphi X, V) - \eta(X)\phi(V, X)$;
$G_1 - K-4)$ $(\nabla_X \phi)(X, Y) - (\nabla_{\varphi X} \phi)(\varphi X, Y) - \eta(X)\phi(Y, X)$.

For the first two components, we note that superminimal fibres means that $\nabla_U \varphi = 0$ because of the vanishing of η on the vertical vector fields. The fourth expression vanishes on basic horizontal vector fields because the projected tensors by the submersion down to the base space vanish.

Now, consider expression $G_1 - K - 3)$. Recall that

$$\begin{aligned}(\nabla_X \phi)(X, V) &= g(X, (\nabla_X \varphi V)) \\ &= g(X, \nabla_X \varphi V - \varphi(\nabla_X V)) \\ &= g(X, A_X \varphi V - \varphi A_X V) \\ &= -g(A_X X, \varphi V) - g(A_X \varphi X, V) \\ &= -g(A_X \varphi X, V).\end{aligned}$$

Similarly, $(\nabla_X \phi)(X, V) = g(A_{\varphi X} X, V)$.

Thus,

$$(\nabla_X \phi)(\varphi X, V) - (\nabla_{\varphi X} \phi)(\varphi X, V) = -g(A_X \varphi X + A_{\varphi X} X, V) = 0.$$

Therefore $G_1 - K - 3)$ vanishes and M is G_1-Kenmotsu.

Considering the case of G_1-semi-Kenmotsu; in this proposition, it remains to consider the case where $\delta\phi' = 0$. It is known that, since the fibres are superminimal, they are Kähler and then minimal. With this, and the use of equation (3.2), of Chinea, [3], we then have $\delta\phi = 0$.

□

Proposition 2.16. *Let $\pi : M^{2m+1} \longrightarrow M'^{2m'+1}$ be an almost contact metric submersion of type I. Assume that the base space is generalized Kenmotsu, quasi Kenmotsu or semi-Kenmotsu normal and the fibres are superminimal. If $(\nabla_X \varphi)V = 0$, then the total space inherits the structure of the base space.*

Proof. Note that $(\nabla_X \varphi)V = 0$, is the almost contact analogue of the Watson criterion [15]. There are six expressions which must vanish to prove that the total space inherits the structure of the base space. We have

GeK-1) $(\nabla_U \phi)(V, W) - (\nabla_{\varphi U} \phi)(\varphi V, W) - \eta(V)\phi(W, U)$;
GeK-2) $(\nabla_U \phi)(V, X) - (\nabla_{\varphi U} \phi)(\varphi V, X) - \eta(V)\phi(X, U)$;
GeK-3) $(\nabla_U \phi)(X, Y) - (\nabla_{\varphi U} \phi)(\varphi X, Y) - \eta(X)\phi(Y, U)$;
GeK-4) $(\nabla_X \phi)(U, V) - (\nabla_{\varphi X} \phi)(\varphi U, V) - \eta(U)\phi(V, X)$;
GeK-5) $(\nabla_X \phi)(Y, V) - (\nabla_{\varphi X} \phi)(\varphi Y, V) - \eta(Y)\phi(V, X)$;
GeK-6) $(\nabla_X \phi)(Y, Z) - (\nabla_{\varphi X} \phi)(\varphi Y, Z) - \eta(Y)\phi(Z, X)$.

Consider expression GeK-4). It is clear that the condition $(\nabla_X \varphi)U = 0$, applies to $(\nabla_{\varphi X} \phi)(\varphi U, V)$, and $(\nabla_X \phi)(U, V)$. Since $\eta(U)\phi(V, X) = 0$, expression GeK-4) vanishes.

In GeK-5), we have to treat $(\nabla_X \phi)(Y, V)$ and $(\nabla_{\varphi X} \phi)(\varphi Y, V)$.

Recall that $(\nabla_X \phi)(Y, V) = g(Y, (\nabla_X \varphi)V)$ and $(\nabla_{\varphi X} \phi)(\varphi Y, V) = g(\varphi Y, (\nabla_{\varphi X} \varphi)V)$, we can apply the condition $(\nabla_X \varphi)U = 0$, The last expression vanishes on basic horizontal vector fields because the projected tensors by the submersion down to the base space vanish.

We have then proved the case of generalized Kenmotsu manifold.

Considering the case of semi-Kenmotsu normal manifold, it remains to consider the case of $\delta\phi' = 0$ and $d\eta = 0$. Since $\delta\phi' = 0$, we can use equation (2.2) of Chinea to get $\delta\phi = 0$. In the light of Lemma 2.2, since $d\eta' = 0$ then $d\eta = 0$.

The case of quasi Kenmotsu is treated in the similar way following generalized Kenmotsu.

□

Proposition 2.17. *Let $\pi : M^{2m+1} \longrightarrow M'^{2m'+1}$ be an almost contact metric submersion of type I. Assume that the base space is G_2-Kenmotsu or almost trans-Kenmotsu and the fibres are superminimal. Even $(\nabla_X\varphi)V = 0$, then the total space does not inherit the structure of the base space.*

Proof. In Lemma 2.2, it is established that if $d\eta' = 0$ on the base space, then the total space also verifies $d\eta = 0$.

As in the case of generalized Kenmotsu structure, there are six components which must vanish to verify the G_2-Kenmotsu structure on the total space. We have

$G_2 - K{-}1)$ $\mathcal{G}\{(\nabla_U\phi)(V,W) - (\nabla_{\varphi U}\phi)(\varphi V,W) - \eta(U)\phi(V,W)\}$;
$G_2 - K{-}2)$ $\mathcal{G}\{(\nabla_U\phi)(V,X) - (\nabla_{\varphi U}\phi)(\varphi V,X) - \eta(U)\phi(V,X)\}$;
$G_2 - K{-}3)$ $\mathcal{G}\{(\nabla_U\phi)(X,Y) - (\nabla_{\varphi U}\phi)(\varphi X,Y) - \eta(U)\phi(X,Y)\}$;
$G_2 - K{-}4)$ $\mathcal{G}\{(\nabla_X\phi)(U,V) - (\nabla_{\varphi X}\phi)(\varphi U,V) - \eta(X)\phi(U,V)\}$;
$G_2 - K{-}5)$ $\mathcal{G}\{(\nabla_X\phi)(Y,V) - (\nabla_{\varphi X}\phi)(\varphi Y,V) - \eta(X)\phi(Y,V)\}$;
$G_2 - K{-}6)$ $\mathcal{G}\{(\nabla_X\phi)(Y,Z) - (\nabla_{\varphi X}\phi)(\varphi Y,Z) - \eta(X)\phi(Y,Z)\}$.

Since η vanishes on vertical vector fields, and the fibres are superminimal, the three first expressions vanish. $G_2 - K{-}5)$ vanishes because $\phi(Y,V) = 0$ since Y is horizontal and V is vertical; $(\nabla_X\phi)(Y,V) = 0$ by the use of $(\nabla_X\varphi)V$, in the same way, we get $(\nabla_{\varphi X}\phi)(\varphi Y,V) = 0$.

The obstruction to the transfer of the structure to the total space is the expression $G_2 - K{-}4)$. Indeed, in this expression, $\eta(X)\phi(U,V) \neq 0$ because $\phi(U,V) = g(U,\varphi V)$ and $\eta(X) \neq 0$.

In order to prove that the total space, $(M^{2m+1}, g, \varphi, \xi, \eta)$, is almost trans Kenmotsu, the following six expressions must vanish.

ATK-1) $\mathcal{G}\left\{(\nabla_U\phi)(V,W) - \frac{1}{m}\phi(U,V)\delta\phi(\varphi W) - 2\eta(U)\phi(V,W)\right\}$;
ATK-2) $\mathcal{G}\left\{(\nabla_U\phi)(Y,X) - \frac{1}{m}\phi(U,Y)\delta\phi(\varphi X) - 2\eta(U)\phi(V,X)\right\}$;
ATK-3) $\mathcal{G}\left\{(\nabla_U\phi)(Y,X) - \frac{1}{m}\phi(U,Y)\delta\phi(\varphi X) - 2\eta(U)\phi(Y,X)\right\}$;
ATK-4) $\mathcal{G}\left\{(\nabla_X\phi)(U,V) - \frac{1}{m}\phi(X,U)\delta\phi(\varphi V) - 2\eta(X)\phi(U,V)\right\}$;
ATK-5) $\mathcal{G}\left\{(\nabla_X\phi)(Y,V) - \frac{1}{m}\phi(X,Y)\delta\phi(\varphi V) - 2\eta(X)\phi(Y,V)\right\}$;
ATK-6) $\mathcal{G}\left\{(\nabla_X\phi)(Y,Z) - \frac{1}{m}\phi(X,Y)\delta\phi(\varphi Z) - 2\eta(X)\phi(Y,Z)\right\}$.

Since $\phi(U,V) \neq 0$, it is clear that ATK-1), ATK-2) and ATK-4) cannot vanish. So, the total space does not inherit the structure of the base space.

Considering ATK-5), we have $\phi(X,Y)\delta\phi(\varphi V) \neq 0$ which obstructs this expression to vanish.

□

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