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Products of Arithmetic Progressions which are Squares

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Abstract

In this short note, we shall give a result similar to Y. Zhang and T. Cai [5] which states the diophantine equation

$$(x-b)x(x+b)(y-b)y(y+b)=z^2$$

has infinitely many *nontrivial* positive integer solutions (x,y,z) when $b(\geq 2)$ is even. We shall show this diophantine equation also has infinitely many *nontrivial* positive integer solutions when integers b is divisible by a prime $p(\equiv \pm 1 \mod 8)$.

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Introduction

Recently in their paper [5] (2015) , Y. Zhang and T. Cai proved there exists infinitely *nontrivial* positive integer solutions of the diophantine equation

$$(x-b)x(x+b)(y-b)y(y+b)=z^2$$

for even number $b \geq 2$. Here the integer solutions (x,y,z) are called *nontrivial* when $b \nmid x$ or $b \nmid y$ and $0<x-b<x<x+b<y-b<y<y+b$. We note that, for the case $b=1$, K. R. S. Sastry showd the above diophantine equation has infinitely many positive integer solutions (x,y,z) (see for example [3] or [5]). The proof of [5] depends on Sastry's idea when $y=2x-1$ the product of the left-hand side of the above diophantine equation is square if $(x+1)(2x-1)=m^2$ for some integer m. Here we shall use the fact that any prime $p \equiv \pm 1 \mod 8$ completely decomposes in $\mathbb{Q}(\sqrt{2})$. Let $p \equiv \pm 1 \mod 8$ and suppose $p|b$. In

the following, we shall verify the above diophantine equation have infinitely many nontrivial positive integer solutions for any such b. We note the problem asking when the product of two or more disjoint blocks of consecutive integers are power of integers was originally stated by P. Erdös and R. L. Graham in [2].

1 Proof of Main Theorem

Since the class number $\mathbb{Q}(\sqrt{2})$ is one and $\left(\frac{2}{p}\right) = 1 \iff p \equiv \pm 1 \mod 8$, the following norm equation has infinitely many positive integer solutions,

$$x^2 - 2y^2 = \pm p.$$

Let (a_0, b_0) be the positive integer solutions $a_0^2 - 2b_0^2 = p$ and put $a_1 = a_0 + 2b_0$ and $b_1 = a_0 + b_0$. Define the binary recurrence sequences $\{a_n\}$ and $\{b_n\}$ by putting

$$a_{n+1} = 2a_n + a_{n-1}, b_{n+1} = 2b_n + b_{n-1} \text{ for } n \geq 1.$$

Then (a_n, b_n) are the positive integer solutions of $x^2 - 2y^2 = (-1)^n p$. Since $(a_{2n} + b_{2n}\sqrt{2})^2 = a_{2n}^2 + 2b_{2n}^2 + 2a_{2n}b_{2n}\sqrt{2}$, we shall put $x = 3b_{2n}^2 + p$ and $y = 2x + p = 3(2b_{2n}^2 + p) = 3a_{2n}^2$. Then except for first few (a_n, b_n) the condition $0 < x - p < x < p < y - p < y < y + p$ is always satisfied. Moreover the direct calculation shows that (x, y) satisfy the following equality

$$(x-p)x(x+p)(y-p)y(y+p) = 2^2 \cdot 3^2 x^2 (x+p)^2 (a_{2n}b_{2n})^2,$$

where $p \not| xy$. Multiplying the both sides of the above equality by $(b/p)^6$, one shall obtain infinitely many nontrivial positive integer solutions of the diophantine equation

$$(X-b)X(X+b)(Y-b)Y(Y+b) = Z^2,$$

where $X = bx/p, Y = by/p$ and $Z = 6(b/p)^3 x(x+p)a_{2n}b_{2n}$.
In the same way as above, (a_{2n+1}, b_{2n+1}) is the positive integer solutions $x^2 - 2y^2 = -p$. Now put $x = 3b_{2n+1}^2 - p$ and $y = 2x - p = 3(2b_{2n+1}^2 - p) = 3a_{2n+1}^2$. Then similarly

$$(x-p)x(x+p)(y-p)y(y+p) = 2^2 \cdot 3^2 x^2 (x-p)^2 (a_{2n+1}b_{2n+1})^2,$$

where $p \not| xy$. In the same way as above, using these sequences of integers, one can also obtain infinitely many nontrivial positive integer solutions for any odd(or even) b which is divisible by p.

Theorem 1.1. *Let b be a positive integer with a prime divisor $p \equiv \pm 1 \mod 8$. Then there exists infinitely many nontrivial positive integer solutions (x, y, z) of the diophantine equation*

$$(x - b)x(x + b)(y - b)y(y + b) = z^2.$$

Remark. If we assume a_0, b_0 are the minimal postive integer solution of the norm equation $x^2 - 2y^2 = p$ with the condition $0 < 2b_0 < a_0$. Since the ideal (p) of $\mathbb{Q}(\sqrt{2})$ decomposes into $(p) = \wp\bar{\wp}$, where $\wp = (a_0 + b_0\sqrt{2})$. Then the conjugate ideal $\bar{\wp} = (a_0 - b_0\sqrt{2}) = ((a_0 - b_0\sqrt{2})(1 + \sqrt{2})) = (a_0 - 2b_0 + (a_0 - b_0)\sqrt{2})$. Put $c_1 = a_0 - 2b_0$ and $d_1 = a_0 - b_0$. Then, from the assumption $0 < 2b_0 < a_0$, (c_1, d_1) are the positive integer solutions of the norm equation $x^2 - 2y^2 = -p$. Put $c_2 = c_1 + 2d_1$ and $d_2 = c_1 + d_1$ and define the binary recurrence sequences c_n and d_n by putting

$$c_{n+1} = 2c_n + c_{n-1}, d_{n+1} = 2d_n + d_{n-1} \text{ for } n \geq 2.$$

Then (c_n, d_n) are also positive integer solutions of $x^2 - 2y^2 = (-1)^n p$. Therefore in the same way as above, we can obtain other infinite series of nontrivial positive solutions of the same diophantine equation

$$(x - b)x(x + b)(y - b)y(y + b) = z^2.$$

Example. Consider the case $b = 7$. From the facts $3^2 - 2 = 7$ and $(3 + \sqrt{2})(1 + \sqrt{2}) = 5 + 4\sqrt{2}$, binary recurrence sequences $\{a_n\}$ and $\{b_n\}$ are defined by

$$a_{n+1} = 2a_n + a_{n-1}, b_{n+1} = 2b_n + b_{n-1},$$

with initial terms $a_0 = 3, a_1 = 5$ and $b_0 = 1, b_1 = 4$. Similarly binary recurrence sequences $\{c_n\}$ and $\{d_n\}$ is defined by

$$c_{n+1} = 2c_n + c_{n-1}, d_{n+1} = 2d_n + d_{n-1},$$

with initial terms $c_1 = 1, c_2 = 5$ and $d_1 = 2, d_2 = 3$. Then the first three non-trivial positive integer solutions are $(x, y, z) = (10, 27, 3060), (34, 75, 125460)$ and $(41, 75, 167280)$. Actually, in the case $(x, y, z) = (10, 27, 3060)$, the direct calculation shows that

$$(3 \times 10 \times 17)(20 \times 27 \times 34) = (2^2 \times 3^2 \times 5 \times 17)^2 = 3060^2.$$

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