$q$-Stokes Phenomenon of a Basic Hypergeometric Series $\phi_1(0; a; q, x)$

By
Yousuke OHYAMA

Department of Mathematical Sciences
Tokushima University
Tokushima 770-8506, JAPAN

E-mail: ohyama@tokushima-u.ac.jp

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Abstract
We show a connection formula of a linear $q$-differential equation satisfied by $\phi_1(0; a; q, x)$. The basic hypergeometric series $\phi_1(0; a; q, x)$ represents the Hahn-Exton $q$-Bessel function. Since the $q$-differential equation has a divergent series solution, a $q$-analogue of the Stokes phenomenon appears. We give a resummation procedure of the divergent series by means of the $q$-Borel-Laplace transformation of order 1/2.

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Introduction
We study the following $q$-difference equation

$$ay(qz) + [z - (a + q)]y(z) + qy(z/q) = 0,$$

which has a solution $y(z) = \phi_1(0; a; q, z)$. We assume that $a \neq 0$. The basic hypergeometric series $\phi_1(0; a; q, z)$ is related to the Hahn-Exton $q$-Bessel function [15], one of the three different types of Jackson’s $q$-analogue of the Bessel function. We solve the connection problem of (1), which gives relations between solutions around the origin and solutions around the infinity. Since (1) has a solution represented by a divergent power series around the infinity, a $q$-analogue of the Stokes phenomenon appears when we give a resummation of the divergent series. We show a resummation of the divergent solution by means of the $q$-Borel-Laplace transformation of order 1/2, which is studied by Dreyfus and Eloy [1, 2].
It is known that there exist three different types of \( q \)-analogues of the Bessel function. Jackson defines his first \( q \)-analogue of the Bessel functions in [6], and the second \( q \)-Bessel function is introduced in [7]. Following the modern notation by Ismail [5], we denote

\[
J_1(x; q) = \left( \frac{q^{x+1} \cdot q}{(q; q)\infty} \right) \frac{x}{2} \cdot \phi_1 \left( 0, 0; q^{x+1}; q, -\frac{x^2}{4} \right),
\]
\[
J_2(x; q) = \left( \frac{q^{x+1} \cdot q}{(q; q)\infty} \right) \frac{x}{2} \cdot \phi_1 \left( -q^{x+1}; q, q^{x+1} x^2 \right),
\]
\[
J_3(x; q) = \left( \frac{q^{2\nu+2} \cdot q^2}{(q^2; q^2)\infty} \right) x^{\nu+1} \cdot \phi_1 \left( 0; q^{2\nu+2}; q^2; x^2 q^2 \right).
\]

Since the third one is found by Hahn [4] and Exton [3], it is called the Hahn-Exton \( q \)-Bessel function, which satisfies the \( q \)-difference equation

\[
(f(x) - q^{-\nu} (x^2 q^2 - 1 - q^{2\nu}) f(x q) + f(x)) = 0. \tag{2}
\]

Any linear \( q \)-difference equation has two singular points, the origin and the infinity. Local solutions around each singular point are represented by a product of theta functions and a formal power series. A connection problem of a linear \( q \)-difference equation is to give a relation between the system of local solutions at the origin and the infinity. When the power series is divergent, the \( q \)-Stokes phenomenon appears by a resummation procedure.

A connection formula of the first Jackson \( q \)-Bessel function is shown by Zhang [19]. Since the second Jackson \( q \)-Bessel function is related to the first Jackson \( q \)-Bessel function

\[
J_2(x; q) = (-x^2 / 4; q)\infty \cdot J_1(x; q),
\]

the connection formula of \( J_2(x; q) \) follows from the connection formula of the first Jackson \( q \)-Bessel function.

But the connection problem for the third Jackson \( q \)-Bessel function has not been solved completely. Solutions of the Hahn-Bessel equation (2) has two independent solutions represented by \( J_3(x; q) \) around the origin. One local solution around the infinity is represented by a convergent power series, and the other is represented by a divergent power series. The asymptotic behavior of \( J_3(x; q) \) around the infinity is studied by Olde Daalhuis [13], but the connection problem is not treated. One connection formula only for the convergent series around the infinity has been shown by Morita [11]. But the \( q \)-Stokes phenomenon of the divergent power series solution is not studied.

In section two we show a \( q \)-difference equation satisfied by \( \phi_1(0; a; q, x) \). In section three we review the \( q \)-Borel transformation and \( q \)-Laplace transformation. In section four we give a resummation of the divergent solution of (1). For divergent power series which satisfy \( q \)-difference equations, the \( q \)-Borel-Laplace
transformation is a powerful tool to give a $q$-summation procedure [14]. The Newton diagram of (1) has two segments at the infinity. The slopes of two segments are 1 and $-1$. Since the difference of the two slopes are two, we need the $q$-Borel resummation of order 1/2 [1, 2]. For $q$-difference linear equations, the Stokes region is not an angle domain, but an open dense set $\mathbb{C}^* \setminus \lambda q^{\mathbb{Z}}$ for $\lambda \in \mathbb{C}^*$ [18, 14]. By using the $q$-Borel-Laplace resummation method of order 1/2, we show the $q$-Stokes phenomenon of the divergent series solution of (1) and the $q$-Stokes region is not outside of a $q$-spiral $\lambda q^{\mathbb{Z}}$ but outside of a $\sqrt{q}$-spiral $\lambda \sqrt{q}^{\mathbb{Z}}$. We remark that a Borel transformation for $q$-series is also studied by Jackson [8].

In section five we show a connection formula for the convergent solution of (1) around the infinity. This formula is essentially shown in [11]. Thus we obtain a complete connection formula of (1).

In the case $a = -q$, (1) reduces to the $q$-Airy equation studied by Hamamoto, Kajiwara and Witte [10]. The $q$-Stokes phenomenon of the $q$-Airy equation is also studied by Morita [12]. Our results contains the connection formula for the $q$-Airy equation.

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1 Notations and Preliminary

In the following we assume that $q \in \mathbb{C}^*$ and $0 < |q| < 1$. For $n = 0, 1, 2, \ldots$, we set the $q$-shifted factorial

$$(a; q)_n = \prod_{j=0}^{n-1} (1 - aq^j), \quad (a; q)_\infty = \prod_{j=0}^{\infty} (1 - aq^j).$$

We set $(a_1, a_2, \ldots, a_m; q)_n = \prod_{j=1}^{m} (a_j; q)_n$ for $n = 0, 1, 2, \ldots$ or $n = \infty$.

We set the theta function

$$\theta_q(x) := \theta(x) = \sum_{k \in \mathbb{Z}} q^{k(k-1)/2} x^k = (q, -x, -q/x; q)_\infty.$$ 

The theta function satisfies

$$\theta(q^k x) = q^{-k(k-1)/2} x^{-k} \theta(x) \quad (k \in \mathbb{Z}),$$
$$x \theta(1/x) = \theta(x), \quad \theta(1/x) = \theta(qx).$$

It is easy to show the following lemma on relations between different bases $q$. 

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It is easy to show the following lemma on relations between different bases $q$. 

Lemma 1. We have
\[
(x; q)_\infty = (x; q^2)_\infty (xq; q^2)_\infty,
\]
\[
(x; q)_\infty (-x; q)_\infty = (x^2; q^2)_\infty,
\]
\[
(q^2; q^2)_\infty \theta_q(x) = (q; q^2)_\infty \theta_{q^2}(xq),
\]
\[
(-q; q)_\infty \theta_q(x) \theta_q(-x) = (q; q)_\infty \theta_{q^2}(-x^2).
\]

1.1 Transformation of \(q\)-difference equation

The \(q\)-difference operator \(\sigma_q\) is given by \(\sigma_q[f(t)] = f(tq)\). We use the following lemma frequently in this paper. The proof is evident.

Lemma 2. We transform a second order \(q\)-difference equation
\[
[a(z)\sigma_q + b(z) + c(z)\sigma_q^{-1}] y(z) = 0.
\]

(1) We set \(t = 1/z\) and \(v(t) = y(1/t)\). Then \(v(t)\) satisfies
\[
[c(1/t)\sigma_q + b(1/t) + a(1/t)\sigma_q^{-1}] v(t) = 0.
\]

(2) We set \(y(z) = \theta(rz)y_1(z)\). Then \(y_1(z)\) satisfies
\[
\left[\frac{a(z)}{rz} \sigma_q + b(z) + \frac{rzc(z)}{q} \sigma_q^{-1}\right] y_1(z) = 0.
\]

(3) We set \(y(z) = (rz; q)_\infty y_2(z)\). Then \(y_2(z)\) satisfies
\[
\left[\frac{a(z)}{1-rz} \sigma_q + b(z) + (1-rz/q)c(z)\sigma_q^{-1}\right] y_2(z) = 0.
\]

1.2 Basic hypergeometric series

The basic hypergeometric series [9] is defined by
\[
r\phi_s(a_1, \ldots, a_r; b_1, \ldots, b_s; q, x) := \sum_{n \geq 0} \frac{(a_1, \ldots, a_r; q)_n}{(b_1, \ldots, b_s; q)_n(q; q)_n} \left\{(-1)^n q^{\frac{n(n+1)}{2}}\right\}^{1+s-r} x^n.
\]

Heine’s basic hypergeometric series \(2\phi_1(a, b; c; q, z)\) satisfies the equation
\[
[(c - abqz)\sigma_q^2 - (c + q - (a + b)qz)\sigma_q + q(1 - z)] 2\phi_1(a, b; c; q, z) = 0.
\]

A connection formula of \(2\phi_1(a, b; c; q, z)\) is shown by Thomaes [16] and Watson [17]:
\[
2\phi_1(a, b; c; q; x) = \frac{(b, c/a; q)_\infty \theta(-ax)}{(c, b/a; q)_\infty \theta(-x)} 2\phi_1(a, aq/c; aq/b; q, /cq/abx)
+ \frac{(a, c/b; q)_\infty \theta(-bx)}{(c, a/b; q)_\infty \theta(-x)} 2\phi_1(b, bq/c; bq/a; q, cq/abx).
\]
It is known that there exist many relations between hypergeometric series. The relation
\[
\phi_1(-; a^2 q^2; 2a^2 q^2) = (x; q)_\infty \cdot \phi_1(a, -a; a^2; q, x)
\] (4)
is shown in [19].

### 1.3 Formal $q$-Borel transformation

We review the $q$-Borel transformation and the $q$-Laplace transformation. See [14, 18, 20] for detail.

The $q$-Borel transformation $B_q^\pm : \mathbb{C}[t] \to \mathbb{C}[\tau]$ is defined by
\[
B_q^\pm \left[ \sum_{n=0}^\infty a_n t^n \right] := \sum_{n=0}^\infty a_n q^{\pm n(n-1)/2} \tau^n.
\]

In usual we identify a germ of holomorphic functions at the origin $\mathcal{O}_{\mathbb{C},0}$ as a subset of $\mathbb{C}[t]$. As a linear operator on $\mathbb{C}[t]$, we have
\[
B_q^\pm (t^m \sigma_q^n f) = q^{\pm m(m-1)/2} \sigma_q^{n+m} B_q^\pm (f).
\]

The $q$-Laplace transform of $\varphi(\tau)$ is given by the Jackson integral
\[
L_{q,1}^{[\lambda]} \varphi(t) := \frac{1}{1-q} \int_0^\lambda \frac{\varphi(\tau)}{\theta_q(\tau/x)} d_q \tau = \sum_{n \in \mathbb{Z}} \frac{\varphi(q^n \lambda)}{\theta_q(q^n \lambda/x)}.
\]

When $f(t) \in \mathbb{C}[t]$ is a convergent power series,
\[
L_{q,1}^{[\lambda]} \circ B_q^+(f) = f.
\]

In this sense, $L_{q,1}^{[\lambda]}$ is a formal inverse of $B_q^+$.

The following lemma is useful to calculate the $q$-Laplace transform. We can prove by direct calculations.

**Lemma 3.** 1) Assume that
\[
\varphi(\xi) = \frac{\theta(a \xi)}{\theta(b \xi)} \sum_{m \geq 0} c_m \xi^{-m}.
\]
Then
\[
L_{q,1}^{[\lambda]} \varphi(x) := \frac{\theta(a \lambda) \theta(qax/b \lambda)}{\theta(b \lambda) \theta(qx/b \lambda)} \sum_{m \geq 0} c_m q^{-m(m-1)/2} (b/aq x)^m.
\]

In the case $a = b$, we obtain
\[
L_{q,1}^{[\lambda]} \left[ \sum_{m \geq 0} c_m \xi^{-m} \right] = \sum_{m \geq 0} c_m q^{-(m)(-m-1)/2} x^{-m},
\]
which gives a formal $q$-Borel transformation $B_q$.

2) Assume that

\[ \varphi(\xi) = \frac{\theta(a\xi)}{\theta(b_1\xi)\theta(b_2\xi)} \sum_{m \geq 0} c_m \xi^{-2m}. \]

Then

\[ L_{q;\lambda}^{|\lambda|} \varphi(x) := \frac{\theta_q(a\lambda)\theta_q(aq^2x/b_1b_2\lambda^2)}{\theta_q(b_1\lambda)\theta_q(b_2\lambda)\theta_q(qx/\lambda)} \sum_{m \geq 0} c_m q^{-m(m-1)} \xi b_1b_2/aq^2 x \xi^m. \]

2 $q$-difference equation satisfied by $1\phi_1(0; a; q, z)$

We assume $a \neq 0$. We consider the following $q$-difference equation

\[ ay(qz) + [z - (a + q)]y(z) + qy(z/q) = 0, \quad (5) \]

which has a solution $y(z) = 1\phi_1(0; a; q, z)$. Since the degree of the coefficients of (1) is up to one, we study (1) instead of the Hahn-Exton equation (2).

We set $t = 1/z$ and $v(t) = y(1/t)$. Then $v(t)$ satisfies

\[ qtv(tq) + [1 - (a + q)t]v(t) + atv(t/q) = 0. \quad (6) \]

In the following, we study a connection problem and the $q$-Stokes phenomenon of (6). Since (6) has a divergent series solution around the origin, the $q$-Stokes phenomenon appears when we give a resummation of the divergent series.

Local solutions of (6) around $t = \infty$ are

\[ v_1^{(\infty)}(t) = 1\phi_1(0; a; q, 1/t), \quad v_2^{(\infty)}(t) = \frac{\theta(-qt)}{\theta(-at)} 1\phi_1(0; q^2/a; q, q/at). \]

Local (formal) solutions of (6) around $t = 0$ are

\[ v_1^{(0)}(t) = \theta(-qt) \sum_{m=0}^{\infty} b_m t^m, \quad v_2^{(0)}(t) = \frac{1}{\theta(-at)} \sum_{m=0}^{\infty} c_m t^m. \]

We assume that $b_0 = 1$ and $c_0 = 1$. Here $u_1(t) = \sum b_m t^m$ is divergent and $u_2(t) = \sum c_m t^m$ is convergent. The $q$-Borel transforms of $u_1(t)$ and $u_2(t)$ are given by

\[ B_q^+(u_1)(\tau) = (-a\tau; q)_{\infty} (-q\tau; q)_{\infty}, \]
\[ B_q^-(u_2)(\tau) = \frac{1}{(-q^2\tau; q)_{\infty} (-aq\tau; q)_{\infty}}. \quad (7) \]
3 $q$-Stokes phenomenon

We give a resummation procedure of the divergent power series $u_1(t)$ by the $q$-Laplace transformation and study the $q$-Stokes phenomenon. We set $v(t) = \theta(-qt)u(t)$ in (6). Then $u(t)$ satisfies

$$\{\sigma_q - [1 - (a + q)t] + at^2\sigma_q^{-1}\}u(t) = 0. \quad (8)$$

The series $u_1(t)$ is a unique formal power series solution of (8) around the origin with $b_0 = 1$.

The $q$-Borel transform of $u_1(t)$ is given by

$$B^+_q(u_1)(\tau) = (-a\tau, -qt;q)_\infty.$$

But the $q$-Laplace transform of $(-a\tau, -qt;q)_\infty$ is divergent. We apply a $q$-analogue of Borel transform of order $1/2$ studied in [1, 2] in order to obtain a resummation of $u_1(t)$.

We set $p^2 = q$. We consider the $p$-Borel-Laplace transform of $u_1(t)$

$$f_p(t, \lambda) = L^{[\lambda]}_{p;1} \circ B^+_p(u_1)(t).$$

The two choices of $p$ give the different $p$-Borel-Laplace transforms. Since $p^2 = q$, $L^{[\lambda]}_{p;1}$ is considered as the $p$-Borel-Laplace transform of order $1/2$.

Our main result is as follows.

**Theorem 4.** The $p$-Borel-Laplace transform $f_p(t, \lambda)$ is a meromorphic function on $\mathbb{C}^*$ and has at most a simple pole on $t = -\lambda p^2$:

$$f_p(t, \lambda) = \frac{\theta_q(\lambda)\theta_q(ap\lambda)}{(q/a;q)_\infty \theta_q(-a\lambda^2) \theta_q(pt/\lambda)\theta_q(qt/\lambda)} 1\phi_1(0; a, 1/t)$$

$$+ \frac{\theta_q(q\lambda)\theta_q(qp\lambda)}{(a/q;q)_\infty \theta_q(-a\lambda^2) \theta_q(pt/\lambda)\theta_q(qt/\lambda)} 1\phi_1(0; q^2/a, q, q/at).$$

**Proof.** The divergent series $u_1(t)$ satisfies the $p$-difference equation

$$\{\sigma_p^2 - [1 - (a + p^2)t] + at^2\sigma_p^{-2}\}u_1(t) = 0.$$

The $p$-Borel transform of $\varphi(\tau) = B^+_p(u_1)(\tau)$ satisfies

$$\{\sigma_p^2 + (a + p^2)\tau \sigma_p - (1 - ap^2)\} \varphi(\tau) = 0. \quad (9)$$

The power series $\varphi(\tau)$ give a unique holomorphic solution around the origin of (9) with $\varphi(0) = 0$. We set $c^2 = ap$. Then $g(\tau) = (-ct, p)_\infty \varphi(\tau)$ satisfies

$$\{(1 + cpr)\sigma_p^2 + (c^2/p + p^2) \tau \sigma_p - (1 - ct)\} g(\tau) = 0,$$
which has a solution \( g(\tau) = 2\phi_1 \left(-c/p, -p^2/c; -p, p, c\tau\right) \). Therefore we have

\[
\varphi(\tau) = \frac{1}{(-c\tau; p)_{\infty}} 2\phi_1 \left(-c/p, -p^2/c; -p, p, c\tau\right).
\]

We study the asymptotic behavior of \( \varphi(\tau) \) around the infinity. It is evident that

\[
\frac{1}{(-c\tau; p)_{\infty}} = \frac{1}{\theta_p(-c\tau; p)}(-p/c; p)_{\infty}.
\]

By the connection formula (3), we have

\[
2\phi_1 \left(-c/p, -p^2/c; -p, p, c\tau\right) = \frac{(p^2/c, -p^2/c; p)_{\infty}}{(-p, p^3/c^2; p)_{\infty}} \theta_p(c^2\tau/p)\theta_p(-c\tau) 2\phi_1 \left(c/p, -c/p; c^2/p^2, p, -p/c\tau\right) + \frac{(c/p, -c/p; p)_{\infty}}{(-p, c^2/p^3; p)_{\infty}} \theta_p(p^2\tau) 2\phi_1 \left(p^2/c, -p^2/c; c^4/p^2, p, -p/c\tau\right).
\]

By (4), we have

\[
(-p/c\tau; p)_{\infty} 2\phi_1 \left(c/p, -c/p; c^2/p^2; p, -p/c\tau\right) = 0\phi_1 \left(-c^2/p^2, p^2/p, \tau^2\right).
\]

\[
(-p/c\tau; p)_{\infty} 2\phi_1 \left(p^2/c, -p^2/c; c^4/p^2; p, -p/c\tau\right) = 0\phi_1 \left(-p^5/c^2, p^2, p^7/c^4\tau^2\right).
\]

Therefore the behavior of \( \varphi(\tau) \) at the infinity is as follows.

\[
\varphi(\tau) = \frac{(p, p^2/c, -p^2/c; p, c^2; p, -p/c\tau)}{(-p, c^2/p^3; p)_{\infty}} \theta_p(c^2\tau/p)\theta_p(-c\tau) \theta_p(-c\tau)\theta_p(p/c\tau) 2\phi_1 \left(-c^2/p^2, p, \tau^2\right) + \frac{(p, c/p, -c/p; p)_{\infty}}{(-p, c^2/p^3; p)_{\infty}} \theta_p(p^2\tau) 2\phi_1 \left(-p^5/c^2, p^2, p^7/c^4\tau^2\right).
\]

We calculate the \( p \)-Laplace transform \( f_p(t, \lambda) \) of \( \varphi(\tau) \) by Lemma 3:

\[
f_p(t, \lambda) = \frac{(p, p^2/c, -p^2/c; p, c^2; p, -p/c\tau)}{(-p, c^2/p^3; p)_{\infty}} \theta_p(c^2\lambda/p)\theta_p(-c\lambda)\theta_p(pt/\lambda) 1\phi_1 \left(0; c^2/p^2, p, 1/t\right) + \frac{(p, c/p, -c/p; p)_{\infty}}{(-p, c^2/p^3; p)_{\infty}} \theta_p(p^2\lambda)\theta_p(-p^4t/c^2\lambda^2) 1\phi_1 \left(0; p^5/c^2, p^2, p^3/c^2t\right).
\]

By Lemma 1, we have

\[
\frac{1}{(-p; p)_{\infty}} \frac{1}{\theta_p(c\lambda)\theta_p(-c\lambda)} = \frac{1}{\theta_p(-c^2\lambda^2)} = \frac{1}{\theta_q(-ap\lambda^2)}.
\]

\[
\frac{1}{(p^2/c, -p^2/c; p)_{\infty}} = \frac{1}{(-p^2/c^2; p)_{\infty}} = \frac{1}{(p^2/c^2; p)_{\infty}} = \frac{1}{(p^2/c^2; p)_{\infty}} = \frac{1}{(q/\alpha; q)_{\infty}}.
\]

And

\[
\frac{\theta_p(a\lambda)\theta_q(-p/\lambda^2)}{\theta_p(-a\lambda^2)} = \frac{\theta_q(a\lambda)\theta_q(ap\lambda)\theta_q(-p/\lambda^2)}{\theta_q(-a\lambda^2)\theta_q(pt/\lambda)\theta_q(qt/\lambda)}.
\]

Applying Lemma 1 to the second term, we obtain Theorem 4.

\[\square\]
4 Connection formula of convergent series

We show a connection formula between \( v_2^{(0)}(t) \) and solutions of (6) around the infinity.

**Theorem 5.** The solution \( v_2^{(0)}(t) \) is written by the sum of \( v_1^{(\infty)}(t) \) and \( v_2^{(\infty)}(t) \) on \( t \in \mathbb{C}^* \):

\[
v_2^{(0)}(t) = \frac{1}{(q; q)_\infty(q/a; q)_\infty} v_1^{(\infty)}(t) + \frac{q}{a \cdot (q; q)_\infty(a/q; q)_\infty} v_2^{(\infty)}(t).
\]

**Remark.** This relation is essentially obtained by Morita [11].

Proof. By (7), \( u_2(t) \) has an integral representation

\[
u_2(t) = \frac{1}{2\pi i} \int_{|\tau| = \varepsilon} \frac{1}{(-q^2 t; q)_\infty(-aq t; q)_\infty} \frac{\theta_q(t/\tau)}{\tau} \, d\tau,
\]

by the residue calculus around the origin. Here \( \varepsilon \) is sufficiently small so that \((-q^2 t; q)_\infty(-aq t; q)_\infty\) does not have zeros in \( |\tau| \leq \varepsilon \).

If we take \( R \) so that the circle \( |z| = R \) does not pass through the poles, we have

\[
\frac{1}{2\pi i} \int_{|\tau| = R} \frac{1}{(-q^2 t; q)_\infty(-aq t; q)_\infty} \frac{\theta_q(t/\tau)}{\tau} \, d\tau \to 0
\]

when \( R \to \infty \). Therefore

\[
u_2(t) = -\sum_{n=0}^{\infty} \text{Res} \left\{ \frac{1}{(-q^2 t; q)_\infty(-aq t; q)_\infty} \frac{\theta_q(t/\tau)}{\tau} : \tau = -q^{-n-2} \right\}
\]

\[-\sum_{n=0}^{\infty} \text{Res} \left\{ \frac{1}{(-q^2 t; q)_\infty(-aq t; q)_\infty} \frac{\theta_q(t/\tau)}{\tau} : \tau = -q^{-n-1}/a \right\}.
\]

We can calculate the residues by the following lemma [18].

**Lemma 6.** We assume that \( b, c \in \mathbb{C}^* \), \( c \notin q^Z \) and \( n = 0, 1, 2, 3, \ldots \). Then we have

\[
\text{Res} \left\{ \frac{1}{(b z; q)_\infty} \frac{dz}{z} : z = q^{-n}/b \right\} = \frac{(-1)^{-n+1} q^{n(n+1)/2}}{(q; q)_\infty(q; q)_n},
\]

\[
\theta_q(bq^n t) = q^{-n(n-1)/2} b^{-n} t^{-n} \theta_q(bt),
\]

\[
\frac{1}{(c q^{-n}; q)_\infty} = \frac{(-c)^{-n} q^{n(n+1)/2}}{(q; q)_\infty(q/c; q)_n}.
\]

By the lemma above we have

\[
u_2(t) = \frac{\theta_q(-q^2 t)}{(q; q)_\infty(a/q; q)_\infty} \phi_1(0; a/q; q; q/\alpha t) + \frac{\theta_q(-aq t)}{(q; q)_\infty(a/q; q)_\infty} \phi_1(0; a; q, 1/t).
\]

Since \( u_2(t) = \theta(-aq t) v_2^{(0)}(t) \), we obtain Theorem 5. \( \square \)
5 Summary

We have shown a connection formula of a second order $q$-difference equation whose solution is represented by $\phi_1(0; a; q, t)$:

$$qtv(tq) + [1 - (a + q)t]v(t) + atv(t/q) = 0.$$  

Local solutions around $t = \infty$ are

$$v_1^{(\infty)}(t) = \phi_1(0; a; q, 1/t), \quad v_2^{(\infty)}(t) = \frac{\theta(-qt)}{\theta(-at)} \phi_1(0; q^2/a; q, q/at).$$

Local (formal) solutions around $t = 0$ are

$$v_1^{(0)}(t) = \theta(-qt) \sum_{m=0}^{\infty} b_m t^m, \quad v_2^{(0)}(t) = \frac{1}{\theta(-at)} \sum_{m=0}^{\infty} c_m t^m.$$

Here $\sum b_m t^m$ is divergent, $\sum c_m t^m$ is convergent. We assume that $b_0 = 1, c_0 = 1$. We set $\tilde{v}_1^{(0)}(t, \lambda; p) = \theta(-qt) f_p(t, \lambda)$ for $p^2 = q$. Here $f_p(t, \lambda)$ is a resummation

$$f_p(t, \lambda) = L_{p;1}^{[\lambda]} \circ B_p^+ \left[ \sum_{m=0}^{\infty} b_m t^m \right].$$

**Theorem 7.** The connection formulae between $v_1^{(0)}(t, \lambda; p), v_2^{(0)}(t)$ and $v_1^{(\infty)}(t), v_2^{(\infty)}(t)$ are given as follows.

$$\tilde{v}_1^{(0)}(t, \lambda; p) = \frac{\theta_q(\lambda\lambda)\theta_q(ap\lambda)}{(q/a; q)_{\infty} \theta_q(-ap\lambda^2)} \theta_q(pt/\lambda) \theta_p(qt/\lambda) v_1^{(\infty)}(t)$$

$$+ \frac{\theta_q(q\lambda)\theta_q(qp\lambda)}{(a/q; q)_{\infty} \theta_q(-ap\lambda^2)} \theta_q(pt/\lambda) \theta_p(qt/\lambda) v_2^{(\infty)}(t),$$

$$v_2^{(0)}(t) = \frac{1}{(q; q)_{\infty} (q/a; q)_{\infty}} v_1^{(\infty)}(t) + \frac{q}{a \cdot (q; q)_{\infty} (a/q; q)_{\infty}} v_2^{(\infty)}(t).$$

The second connection formula is already shown by Morita [11]. The case $a = -q$ is obtained in [12]. A connection formula of the Hahn-Exton $q$-Bessel equation is derived from the theorem above by simple calculations.

The $q$-Laplace transform of order 1/2 is shown in [1, 2] is necessary to determine the $q$-Stokes coefficients. Our results is the first example to calculate the $q$-Stokes coefficients when the slope of the Newton diagram is two in $q$-difference equations. Our method would be useful to study the $q$-Stokes phenomenon of other $q$-difference equations with slopes higher than two.

References


[16] Thomae J.; Ueber die Functionen welche durch Reihen von der Form dargestellt werden \( 1 + \frac{1}{q} x + \frac{1}{q^2} x^2 + \frac{1}{q^3} x^3 + \frac{1}{q^4} x^4 + \frac{1}{q^5} x^5 + \frac{1}{q^6} x^6 + \cdots \), J. Reine Angew. Math. 87 (1879), 26–73.

