New Proof of Plancherel's Theorem

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Abstract

In this paper, we study the new proof of Plancherel's Theorem for the Fourier transformation of $L^2(\boldsymbol{R}^d)$. Here we asuume $d \geq 1$. We use the method of orthogonal measure and orthogonal integral which is the generalization of Kato [3].

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Introduction

In this paper, we give the new proof of the following Plancherel's Theorem. This paper is the English version of Ito [2], section 4.2.

Main Theorem (Plancherel's Theorem) *Assume $d \geq 1$. The Fourier transformation $\mathcal{F}$ of $L^2 = L^2(\boldsymbol{R}^d)$ is a unitary transformation of L^2. Namely we have the equality*

$$\|\mathcal{F}f\| = \|f\|$$

for an arbitrary $f \in L^2$. Here $\|\cdot\|$ denotes the L^2-norm.

We prove this theorem in the case $d \geq 1$ by using the method of orthogonal measure and orthogonal integral mentioned in section 2.

This is the generalization of the proof of Kato [3], p.130 in the case $d = 1$. Thereby, we clarify the true meaning of Kato's method. Kato proved this theorem by using only the calculation of integrals. In fact, the theorem is proved by using only the definition of integrals and the properties of defining functions of bounded measurable sets. Thus we need not use the special functions.

Here I show my heartfelt gratitude to my wife Mutuko for her help of typesetting this manuscript.

1 Fourier transformation of L^2-functions

In this section, we define the Fourier transformation of L^2-functions.

Assume that $d \geq 1$ and $\boldsymbol{R}^d$ is the d-dimensional Euclidean space.

$\boldsymbol{R}^d$ is a self-dual space. Thus we identify the dual space of $\boldsymbol{R}^d$ with itself and we denote it as the same symbol $\boldsymbol{R}^d$. For a point $x = {}^t(x_1,\ x_2,\ \cdots,\ x_d)$ in $\boldsymbol{R}^d$ and a point $p = {}^t(p_1,\ p_2,\ \cdots, p_d)$ in its dual space $\boldsymbol{R}^d$, we define the dual inner product by the relation

$$px = (p,\ x) = p_1x_1 + p_2x_2 + \cdots + p_dx_d.$$

Then we define the norms $|x|$ and $|p|$ by the relations

$$|x| = \sqrt{|x_1|^2 + |x_2|^2 + \cdots + |x_d|^2},\ |p| = \sqrt{|p_1|^2 + |p_2|^2 + \cdots + |p_d|^2}.$$

Definition 1.1(Fourier transformation) For $f \in L^2 = L^2(\boldsymbol{R}^d)$, we define the **Fourier transform** $(\mathcal{F}f)(p)$ by the relation

$$(\mathcal{F}f)(p) = \underset{R\to\infty}{\text{l.i.m.}} \frac{1}{(\sqrt{2\pi})^d} \int_{|x|\leq R} f(x)e^{-ipx}dx.$$

In Definition 1.1, the symbol l.i.m. denotes the limit in the mean. Thus we have $(\mathcal{F}f)(p) \in L^2$.

Then we denote $(\mathcal{F}f)(p)$ as

$$(\mathcal{F}f)(p) = \frac{1}{(\sqrt{2\pi})^d} \int_{\boldsymbol{R}^d} f(x)e^{-ipx}dx.$$

2 Orthogonal measure and orthogonal integral

In this section, we define the concept of orthogonal measure and orthogonal integral and study its properties. As for this concept, we refer to Ito [2], chapter 8.

Proposition 2.1 *Assume that* $(\boldsymbol{R}^d,\ \mathcal{M},\ \mu)$ *is the Lebesgue measure space and* $\mathcal{M}_b$ *is the family of all bounded measurable sets in* $\boldsymbol{R}^d$. *If we restrict* μ *on* $\mathcal{M}_b$, *we have the measure space* $(\boldsymbol{R}^d,\ \mathcal{M}_b,\ \mu)$. *Then, assuming that the function* $\chi_E(x)$ *is the defining function of a set* E, *the* L^2*-valued set function* $\chi : E \to \chi_E$ *on* $\mathcal{M}_b$ *is an orthogonal measure on* $(\boldsymbol{R}^d,\ \mathcal{M}_b,\ \mu)$. *Namely we have the following* (1) *and* (2):

(1) *If each pair of a countable sequence* $E_1,\ E_2,\ \cdots$ *of sets of* $\mathcal{M}_b$ *are mutually disjoint and the direct sum* E *is equal to*

$$E = \sum_{j=1}^{\infty} E_j$$

and we have $E \in M_b$, *the equality*

$$\chi_E = \sum_{j=1}^{\infty} \chi_{E_j}$$

holds. Here the series in the right hand side converges in the sense of L^2*-convergence.*

(2) *If we have* $E_1,\ E_2 \in \mathcal{M}_b$, *the equality*

$$(\chi_{E_1},\ \chi_{E_2}) = \mu(E_1 \cap E_2)$$

holds. Here the symbol $(\cdot, \cdot)$ *denotes the inner product of* L^2.

Corollary 2.1 *We use the notation of Proposition* 2.1. *Then we have the following* (1) *and* (2):

(1) *If* $E_1 \cap E_2 = \emptyset$ *for* $E_1,\ E_2 \in \mathcal{M}_b$, χ_{E_1} *and* χ_{E_2} *are orthogonal in* L^2.

(2) *For* $E \in \mathcal{M}_b$, *we have the equality*

$$\mu(E) = \|\chi_E\|^2.$$

Here the symbol $\| \cdot \|$ in the right hand side denotes the norm of L^2.

Theorem 2.1 *Assume that* $(\boldsymbol{R}^d,\ \mathcal{M},\ \mu)$ *is the Lebesgue measure space and* $\mathcal{M}_b$ *is the family of all bounded measurable sets in* $\boldsymbol{R}^d$*. The* L^2*-valued set function* $\chi :\ E \to \chi_E$ *on* $\mathcal{M}_b$ *is an orthogonal measure on* $(\boldsymbol{R}^d,\ \mathcal{M}_b,\ \mu)$*. Now, for* $f \in L^2$*, we define the orthogonal integral of* f

$$\int f(x) d\chi(x)$$

by using the orthogonal measure χ. Then we have the equality

$$f(x) = \int f(x) d\chi(x),\ (x \in \boldsymbol{R}^d).$$

Further, we have the equality

$$\| \int f(x) d\chi(x) \|^2 = \int |f(x)|^2 d\mu(x)$$

for the L^2*-norm.*

Proof We define the orthogonal integral in the following two steps.
(I) The case where $f(x)$ is s simple L^2-function.
Now we assume that $f(x)$ is represented as

$$f(x) = \sum_{j=1}^{\infty} a_j \chi_{E_j}(x),\ (a_j \in \boldsymbol{C},\ j \geq 1),$$

$$\boldsymbol{R}^d = E_1 + E_2 + \cdots,\ (E_j \in \mathcal{M}_b,\ j \geq 1).$$

We define the orthogonal integral by the following relation

$$\int f(x) d\chi(x) = \sum_{j=1}^{\infty} a_j \chi_{E_j}(x).$$

Then we have the equality

$$f(x) = \int f(x) d\chi(x).$$

Further we have the equality

$$\| \int f(x) d\chi(x) \|^2 = \sum_{j=1}^{\infty} |a_j|^2 \| \chi_{E_j}(x) \|^2$$

$$= \sum_{j=1}^{\infty} |a_j|^2 \mu(E_j) = \int |f(x)|^2 d\mu(x)$$

for the L^2-norm.

(II) The case where $f(x)$ is a general L^2-function.

In this case, there exists a sequence of simple L^2-functions $\{f_m\}$ so that f_m converges to f in the sense of L^2-convergence. Then we define the orthogonal integral of f by virtue of the orthogonal measure χ as follows:

$$\int f(x) d\chi(x) = \lim_{m \to \infty} \int f_m(x) d\chi(x).$$

Here the limit in the right hand side is considered in the sense of L^2-convergence. Then we have the equality

$$f(x) = \int f(x) d\chi(x).$$

Further we have the equality

$$\| \int f(x) d\chi(x) \|^2 = \int |f(x)|^2 d\mu(x)$$

for the L^2-norm. //

Assume that E is a bounded measurable set in $\boldsymbol{R}^d$. Then, by defining the Fourier transform $\hat{\chi}_E(p)$ of $\chi_E(x)$ by the relation

$$(\mathcal{F}\chi_E)(p) = \hat{\chi}_E(p),$$

we have $\hat{\chi}_E \in L^2$.

Proposition 2.2 *For every pair $E_1,\ E_2$ of bounded measurable sets in $\boldsymbol{R}^d$, we have the equality*

$$(\hat{\chi}_{E_1},\ \hat{\chi}_{E_2}) = (\chi_{E_1},\ \chi_{E_2}) = \mu(E_1 \cap E_2).$$

Proof We prove this proposition in the following three steps (I), (II), (III).

(I) In the case $d = 1$. Assume that $a < b,\ c < d$. Then we prove the equality

$$(\hat{\chi}_{(a,\, b)},\ \hat{\chi}_{(c,\, d)}) = (\chi_{(a,\, b)},\ \chi_{(c,\, d)}) = \mu((a,\ b) \cap (c,\ d)).$$

As for this proof, we refer to Kato [3].

At first, we have the equality

$$\hat{\chi}_{(a,\,b)}(p) = \frac{1}{\sqrt{2\pi}}\int_a^b e^{-ipx}dx = \frac{1}{\sqrt{2\pi}}\frac{i}{p}(e^{-ibp} - e^{-iap}).$$

Then we have the equality

$$(\hat{\chi}_{(a,\,b)},\ \hat{\chi}_{(c,\,d)}) = \int_{-\infty}^{\infty} \overline{\hat{\chi}_{(a,\,b)}(p)}\hat{\chi}_{(c,\,d)}(p)dp$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}(e^{ibp} - e^{iap})(e^{-idp} - e^{-icp})\frac{dp}{p^2}$$

$$= \frac{1}{\pi}\int_0^{\infty}(\cos(b-d)p + \cos(a-c)p - \cos(b-c)p - \cos(a-d)p)\frac{dp}{p^2}.$$

Here, by using Dirichlet integral

$$\lim_{\lambda\to\infty}\int_0^{\lambda}\frac{\sin\alpha x}{x}dx = \frac{\pi}{2}\operatorname{sign}\alpha$$

for an arbitrary real number α, we have the equality

$$\int_0^{\infty}(1-\cos kp)\frac{dp}{p^2} = \lim_{\lambda\to\infty} k\int_0^{\lambda}\frac{\sin kp}{p}dp = \frac{\pi}{2}|k|.$$

Thus we have the equality

$$(\hat{\chi}_{(a,\,b)},\ \hat{\chi}_{(c,\,d)}) = \frac{1}{2}(|b-c| + |a-d| - |b-d| - |a-c|)$$

$$= \mu((a,\ b)\cap(c,\ d)) = (\chi_{(a,\,b)},\ \chi_{(c,\,d)}).$$

Therefore we proved the equality

$$(\hat{\chi}_{(a,\,b)},\ \hat{\chi}_{(c,\,d)}) = (\chi_{(a,\,b)},\ \chi_{(c,\,d)}) = \mu((a,\ b)\cap(c,\ d)).$$

We remark that this relation holds not only for bounded open intervals but also for any bounded intervals. Therefore this relation holds for any bounded blocks of intervals.

(II) In the case $d \geq 2$, we generalize the relation in (I).

For $a = {}^t(a_1,\ a_2,\ \cdots, a_d),\ b = {}^t(b_1,\ b_2,\ \cdots, b_d) \in \boldsymbol{R}^d$, we denote $a < b$ if the conditions $a_j < b_j,\ (1 \leq j \leq d)$ are satisfied. Then, for $a,\ b \in \boldsymbol{R}^d$ such as $a < b$, we denote the d-dimensional open interval as

$$(a,\ b) = \prod_{j=1}^{d}(a_j,\ b_j)$$

and its defining function as

$$\chi_{(a,\,b)}(x) = \prod_{j=1}^{d} \chi_{(a_j,\,b_j)}(x_j),\ (x = {}^t(x_1,\ x_2,\ \cdots,\ x_d) \in \boldsymbol{R}^d).$$

Then the Fourier transform of $\chi_{(a,\,b)}(x)$ is equal to

$$\hat{\chi}_{(a,\,b)}(p) = \prod_{j=1}^{d} \hat{\chi}_{(a_j,\,b_j)}(p_j),\ (p = {}^t(p_1,\ p_2,\ \cdots,\ p_d) \in \boldsymbol{R}^d).$$

Now assume that $a < b,\ c < d$ hold.
Then we have the equality

$$(\hat{\chi}_{(a,\,b)},\ \hat{\chi}_{(c,\,d)}) = \prod_{j=1}^{d}((\hat{\chi}_{(a_j,\,b_j)},\ \hat{\chi}_{(c_j,\,d_j)})$$

$$= \prod_{j=1}^{d}(\chi_{(a_j,\,b_j)},\ \chi_{(c_j,\,d_j)}) = \prod_{j=1}^{d} \mu((a_j,\ b_j) \cap (c_j,\ d_j)) = \mu((a,\ b) \cap (c,\ d)).$$

We remark that this relation holds not only for bounded open intervals but also for any bounded intervals. Therefore this relation holds for any bounded blocks of intervals.

(III) At last, we prove the equality

$$(\hat{\chi}_{E_1},\ \hat{\chi}_{E_2}) = (\chi_{E_1},\ \chi_{E_2}) = \mu(E_1 \cap E_2)$$

for any $E_1,\ E_2 \in \mathcal{M}_b$.

By virtue of the definition of Lebesgue measure, for $E_1,\ E_2 \in \mathcal{M}_b$, there exist two sequences of bounded blocks of intervals $\{A_n\}$ and $\{B_n\}$ such that we have

$$\mu(E_1 \Delta A_n) \to 0,\ \mu(E_2 \Delta B_n) \to 0.$$

Therefore we have the relations

$$\mu((E_1 \cap E_2)\Delta(A_n \Delta B_n)) \to 0.$$

Here, as for the definition of Lebesgue measure, we refer to Ito [1],

Therefore $\chi_{A_n}(x)$ converges to $\chi_{E_1}(x)$ in measure and $\chi_{B_n}(x)$ converges $\chi_{E_2}(x)$ in measure. Therefore, we have $\chi_{A_n}(x) \to \chi_{E_1}(x)$ and $\chi_{B_n}(x) \to \chi_{E_2}(x)$ in the sense of L^2-convergence. Hence we have the equality

$$(\hat{\chi}_{E_1},\ \hat{\chi}_{E_2}) = \lim_{n\to\infty}(\hat{\chi}_{A_n},\ \hat{\chi}_{B_n}) = \lim_{n\to\infty}(\chi_{A_n},\ \chi_{B_n}) = (\chi_{E_1},\ \chi_{E_2}).$$

Further we have the equality

$$\lim_{n\to\infty}(\chi_{A_n},\ \chi_{B_n}) = \lim_{n\to\infty} \mu(A_n \cap B_n) = \mu(E_1 \cap E_2).$$

Thus we have the equality

$$(\hat{\chi}_{E_1},\ \hat{\chi}_{E_2}) = (\chi_{E_1},\ \chi_{E_2}) = \mu(E_1 \cap E_2)$$

for $E_1,\ E_2 \in \mathcal{M}_b$. //

3 Proof of Plancherel's Theorem

In this section, we prove the Main Theorem. Here the new method of the proof of this theorem is the method of orthogonal measure. In fact we prove the Main Theorem by using the results of section 2. This method is very new. This proof is the completion of the idea of Kato [3].

Now, we prove the Main Theorem in the following two steps.

(I) In the case where $f(x)$ is a simple L^2-function.

Now we assume that $f(x)$ is represented as follows:

$$f(x) = \sum_{j=1}^{\infty} a_j \chi_{E_j}(x),\ (a_j \in \boldsymbol{C},\ j \geq 1),$$

$$\boldsymbol{R}^d = \sum_{j=1}^{\infty} E_j,\ (E_j \in \mathcal{M}_b,\ j \geq 1).$$

Then we have the equality

$$\int |f(x)|^2 dx = \sum_{j=1}^{\infty} |a_j|^2 \mu(E_j) < \infty.$$

The Fourier transform $\mathcal{F}f$ of f is equal to the relation

$$(\mathcal{F}f)(p) = \sum_{j=1}^{\infty} a_j \hat{\chi}_{E_j}(p).$$

Then, by virtue of Proposition 2.2, we have the equality

$$\int |(\mathcal{F}f)(p)|^2 dp = \sum_{j=1}^{\infty} |a_j|^2 \int |\hat{\chi}_{E_j}(p)|^2 dp = \sum_{j=1}^{\infty} |a_j|^2 \mu(E_j) = \int |f(x)|^2 d\mu(x).$$

(II) In the case where $f(x)$ is a general L^2-function.

In this case, there exists a sequence of simple L^2-functions $\{f_m\}$ so that f_m converges to f in the sense of L^2-convergence. Then, by virtue of (I), we have the equality

$$\|\mathcal{F}f_m\| = \|f_m\|,\ (m \geq 1).$$

Further, because $\{f_m\}$ is a Cauchy sequence in L^2, $\{\mathcal{F}f_m\}$ is also a Cauchy sequence in L^2 and we have the equality

$$\mathcal{F}f = \lim_{m\to\infty} \mathcal{F}f_m$$

in the sense of L^2-convergence.

Thus we have $\mathcal{F}f \in L^2$ and the equality

$$\|\mathcal{F}f\| = \|f\|.//$$

References

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