Prescribed Mean Curvature Equations for Functions of Bounded Variation

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Abstract

Let $L(u) = L(u, \nabla u)$ be a functional on $W^{1,1}(\Omega)$ whose formal Euler-Lagrange equation at the critical point u of L is the prescribed mean curvature equation:

$$-\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = g(x, u).$$

Suppose $\mathcal{L}(u) = \mathcal{L}(u, Du)$ is a relaxed functional of $L(u)$, the weakly lower semicontinuous extension of L on the space of functions of bounded variation. How dose the relaxation affect the prescribed mean curvature equation? Instead of an Euler-Lagrange equation, we obtain here the so-called Euler-Lagrange system of equations which the critical points u of $\mathcal{L}$ and their derivatives Du necessarily satisfy.

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Introduction

We are concerned here with a Dirichlet boundary value problem (DBVP) of the prescribed mean curvature equation:

$$\begin{cases} -\mathrm{div}\left(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = \lambda g(x, u) & \text{in } \Omega, \\ u = 0 & \text{on} \Gamma \end{cases} \tag{0.1}$$

Here Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with sufficiently smooth boundary $\Gamma = \partial\Omega$, g is a Carathéodory function and λ a positive real number. The problem is derived (at least formally) as an Euler-Lagrange equation from a variational problem for the functional

$$L_\lambda(u) = \int_\Omega a(\nabla u)dx - \lambda \int_\Omega G(x,u)dx \tag{0.2}$$

on $W_0^{1,1}(\Omega)$, where $a(v) = \sqrt{1+|v|^2} - 1$ on $v \in \mathbb{R}^N$ and $G(x,u) = \int_0^u g(x,s)ds$. However, the principal part $\mathcal{A}_0(u) = \int_\Omega a(\nabla u)dx$, called an area functional, is not lower semicontinuous with respect to the weak topology (or even in L_1-topology) of the space, and neither is L_λ. A standard approach to the variational problem of such a functional is to relax the functional in such a way the resulting functional becomes lower semicontinuous and to seek critical points of the relaxed functional. In the case of L_λ, the relaxed functional $\mathcal{L}_\lambda$ is defined on $BV(\Omega)$, the space of functions of bounded variation on Ω. The existence of a local minimum point and a non-minimal critical point of $\mathcal{L}_\lambda$ has been proven by V. K. Le [11, 12].

We here direct our attention to the existence problem of the Euler-Lagrange equation itself like (0.1) for $\mathcal{L}_\lambda$. How is it expressed if it exists? We cannot use the minimality, of course, to characterize the non-minimal critical points of $\mathcal{L}_\lambda$. For the non-differentiable functional $\mathcal{L}_\lambda$, its critical points cannot be defined as zeros of its derivative, and should be done in some indirect way. V. K. Le defines the critical point of $\mathcal{L}_\lambda$ as a solution of a variational inequality. Therefore it is of interest to obtain some Euler-Lagrange equations like (0.1) which critical points necessarily satisfy. Since the critical point u belongs to $BV(\Omega)$, its distributional derivative Du is a bounded measure. Thus the expected Euler-Lagrange equation will involve the measure Du together with u as unknowns. The measure Du can be divided into some parts which are singular each other. For instance, $Du = D^a u + D^s u$, where $D^a u$ (or $D^s u$) is the absolutely continuous (or singular respectively) part with respect to the N-dimensional Lebesgue measure $dx = d\mathscr{L}^N$. And the Euler-Lagrange equation may be described as a system of some equations, each of those governs one part of Du. In that case, we call it the Euler-Lagrange system and refer to it as the prescribed mean curvature equations of (DBVP) since the system itself governs a function $u \in BV(\Omega)$ regarded as a solution of (DBVP). For the critical point u of $\mathcal{L}_\lambda$ defined as a solution of the variational inequality proposed by Le, we obtain the prescribed mean curvature equations of (DBVP). To state our result more precisely, we give some preliminaries and a short summary of results gotten by V. K. Le.

1 Preliminaries and the main result

In this section, we give some preliminaries together with a short summary of the existence results of the critical points of $\mathcal{L}_\lambda$ proved by Le. And then we state the main theorem of the present paper.

The space of functions of bounded variation on Ω is defined by

$$BV(\Omega) = \{u \in L^1(\Omega) : |Du|_\Omega < +\infty\} \tag{1.3}$$

where

$$|Du|_\Omega = \sup\{\int_\Omega u\mathrm{div}\eta \, dx : \eta = (\eta_1, \cdots, \eta_N) \in C_c^1(\Omega; \mathbb{R}^N), |\eta| \leq 1\} \tag{1.4}$$

is called the total variation of u on Ω. Du is the distributional derivative of u, namely,

$$< Du, \eta >= -\int_\Omega u\mathrm{div}\eta \, dx \qquad \text{for all } \eta \in C_c^\infty(\Omega; \mathbb{R}^N),$$

and it is a $\mathbb{R}^N$-valued Radon measure on Ω if $u \in BV(\Omega)$. $|Du|$ is a positive Radon measure satisfying

$$|Du|(O) \equiv \int_O |Du| = |Du|_O$$

for all open set $O \subset \Omega$. $BV(\Omega)$ is a Banach space with the norm $||u|| = ||u||_{L^1(\Omega)} + |Du|_\Omega$. It is well known that $BV(\Omega)$ is embedded in $L^{1^*}(\Omega)$ ($1^* = \frac{N}{N-1}$) continuously. The function $u \in BV(\Omega)$ has a trace u_Γ which is the boundary value of u on Γ if $u \in C^1(\overline{\Omega})$. The mapping $\gamma : BV(\Omega) \to L^1(\Gamma)$ with $\gamma(u) = u_\Gamma$ is continuous.

The definition of the area functional $\mathcal{A}(u : \Omega) = \int_\Omega a(Du)$ has been introduced by E. Giusti [7].

Definition 1.1. Let U be a bounded domain in $\mathbb{R}^N$. Define an area functional $\mathcal{A}$ on $L^{1^*}(U)$ by

$$\mathcal{A}(u : U) = \sup\{\int_U (\eta_0 + u\mathrm{div}\eta - 1)dx : \hat{\eta} = (\eta_0, \eta) \in C_c^1(U; \mathbb{R} \times \mathbb{R}^N), |\hat{\eta}| \leq 1\} \tag{1.5}$$

where $\eta = (\eta_1, \cdots, \eta_N)$ and $\mathrm{div}\eta = \sum_{i=1}^N \frac{\partial \eta_i}{\partial x_i}$.

Observe that $\mathcal{A} = \mathcal{A}(\cdot : U)$ is a convex functional on $L^{1^*}(U)$ with the effective domain $\mathrm{dom}\mathcal{A} = BV(U)$, i.e., $\mathcal{A}(u : U)$ is finite if and only if $u \in BV(U)$, and is lower semicontinuous in L^1 and thus in L^{1^*}-toplogy.

The relaxation $\mathcal{A}_1$ of $\mathcal{A}_0$ on $W_0^{1,1}(\Omega)$ in the introduction, $\mathcal{A}_0(u) = \int_\Omega a(\nabla u)dx$ for $u \in W_0^{1,1}(\Omega)$, is known as the functional on $BV(\Omega)$ defined by

$$\mathcal{A}_1(u) = \mathcal{A}(u : \Omega) + \int_\Gamma |u_\Gamma| d\mathscr{H}^{N-1} \qquad \text{for } u \in BV(\Omega) \tag{1.6}$$

The integral in the right-hand side, which is the relaxation term for the boundary condition $u = 0$ on Γ, can be eliminated by extending function u by taking the value zero outside Ω. Let $\mathscr{B}$ be a open ball (with center 0) such that $\overline{\Omega} \subset \mathscr{B}$ and put

$$\overline{u} = \begin{cases} u & \text{in } \Omega \\ 0 & \text{in } \mathscr{B} \setminus \Omega \end{cases} ,$$

then $\overline{u} \in BV(\mathscr{B})$ and

$$\mathcal{A}_1(u) = \mathcal{A}(\overline{u} : \mathscr{B}).$$

Setting

$$X = \{u \in BV(\mathscr{B}) : u = 0 \text{ on } \mathscr{B} \setminus \Omega\} ,$$

we can replace (1.6) by

$$\mathcal{A}(u : \mathscr{B}) = \int_{\mathscr{B}} a(Du) \tag{1.7}$$

for $u \in X$. Remark that the above integral is here a formal and convenient expression of the functional $\mathcal{A}(u : \mathscr{B})$ defined in Definition 1.1. We give later another definition of the integral and prove that the above equality holds for $u \in BV(\mathscr{B})$ with compact support (see (2.35)). Since we only deal with functions in X or $BV(\mathscr{B})$ in what follows, we write simply $\mathcal{A}(\cdot) = \mathcal{A}(\cdot : \mathscr{B})$.

We assume the following conditions:

(A. 1) $g : \mathscr{B} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying

$$g(x, \xi) = 0 \qquad \text{for } x \in \mathscr{B} \setminus \Omega.$$

(A. 2) There exists $q \in (1, 1^*)$ such that

$$|g(x, \xi)| \leq d_1 |\xi|^{q-1} + d_2 \quad \text{for a.e. } x \in \mathscr{B}, \text{ all } \xi \in \mathbb{R}$$

with some constant $d_1, \ d_2 > 0$.

Put

$$\mathcal{G}(u) = \int_{\mathscr{B}} G(x, u) dx \tag{1.8}$$

for $u \in L^{1^*}(\mathscr{B})$. By (A. 2), $\mathcal{G}$ is Fréchet differentiable and

$$< \mathcal{G}'(u), v >= \int_{\mathscr{B}} g(x, u) v dx \tag{1.9}$$

for $v \in L^{1^*}(\mathscr{B})$. The relaxation $\mathcal{L}_\lambda$ of L_λ in the introduction is a functional on X denoted by

$$\mathcal{L}_\lambda(u) = \mathcal{A}(u) - \lambda \mathcal{G}(u) \tag{1.10}$$

for $u \in X$.

Under (A. 1), (A. 2) and some additional conditions, the existence of a local minimum point and a non-minimal critical point of functional $\mathcal{L}_\lambda$ has been proven by V. K. Le [11]. Since $\mathcal{A}$ is not differentiable, introducing the "weak slope"(in [5]) instead of the derivative and using the mountain pass argument (in [8]), he proved the existence of the non-minimal critical point of $\mathcal{L}_\lambda$ for small λ, which is, as a result, a solution $u \in X$ of the variational inequality:

$$\mathcal{A}(v) - \mathcal{A}(u) - \lambda \int_{\mathscr{B}} g(x,u)(v-u)dx \geq 0 \qquad \text{for all} \quad v \in X. \tag{1.11}$$

The minimum point u of $\mathcal{L}_\lambda$ also satisfies (1.11). When a pair (λ, u) satisfies (1.11) λ is called an eigenvalue and u its eigenfunction. By using a Ljusternik-Schnirelmann theory for (1.11), Le [12] also obtained infinite sequence of eigenvalues and eigenfunctions.

Since the functional $\mathcal{A}$ is convex on X, the inequality (1.11) implies

$$0 \in \partial\mathcal{A}(u) - \lambda\mathcal{G}'(u) \tag{1.12}$$

where $\partial\mathcal{A}(u)$ is the subdifferential of $\mathcal{A}$ at u. Thus (1.12) or the inequality (1.11) itself can be regard as a weak form of the Euler-Lagrange equation for the critical point of $\mathcal{L}_\lambda$. Our aim is to obtain a more explicit expression.

For $u \in X$, we denote by u_Ω the restriction of u onto Ω, then $u_\Omega \in BV(\Omega)$ and

$$Du = Du_\Omega - u_\Gamma \nu d\mathscr{H}^{N-1}, \tag{1.13}$$

where u_Γ is the trace of u_Ω on Γ, $u_\Gamma = (u_\Omega)_\Gamma$ (see e.g. [1, 2]).

Let $U \subset \mathbb{R}^N$ be open and $v \in BV(U)$, $D^a v$ and $D^s v$ be respectively the absolutely continuous and singular parts of the Radon measure Dv. Denote the density $\frac{D^a v}{d\mathscr{L}^N}$ by $\nabla v \in L^1(U)$, then

$$Dv = D^a v + D^s v = \nabla v dx + D^s v.$$

For $u \in X$, by (1.13), we have

$$\begin{aligned} D^a u =& D^a u_\Omega = \nabla u_\Omega dx, \\ D^s u =& D^s u_\Omega + u_\Gamma \nu d\mathscr{H}^{N-1}, \\ Du =& \nabla u_\Omega dx + D^s u_\Omega - u_\Gamma \nu d\mathscr{H}^{N-1}. \end{aligned}$$

The main result of the present paper ia as follows.

Theorem 1.2. *Let $u \in X$ be a solution of the variational inequality* (1.11), *then there exists $p \in L^\infty(\mathscr{B}; \mathbb{R}^N)$ such that*

$$-\mathrm{div} p = \lambda g(x, u_\Omega) \qquad \textit{in } \Omega, \tag{1.14}$$

$$p = \frac{\nabla u_\Omega}{\sqrt{1+|\nabla u_\Omega|^2}} \qquad \textit{in } \Omega, \tag{1.15}$$

$$p \cdot D^s u_\Omega = |D^s u_\Omega| \qquad \textit{in } \Omega, \tag{1.16}$$

$$(\nu \cdot p) u_\Gamma = -|u_\Gamma| \qquad \textit{on } \Gamma. \tag{1.17}$$

where $\nu \cdot p$ is the weakly-defined trace of the nomal component of p, which lies in $L^\infty(\Gamma)$.

Remark 1.3. In the above theorem, we can eliminate p. By (1.15), $p \in L^\infty$ is obvious. Substituting (1.15) into (1.14), we have the prescribed mean curvature equation on u_Ω:

$$-\mathrm{div}\left(\frac{\nabla u_\Omega}{\sqrt{1+|\nabla u_\Omega|^2}}\right) = \lambda g(x, u_\Omega). \quad \text{in } \Omega \tag{1.18}$$

Substitute (1.15) into (1.16) and (1.17). Then we have

$$\frac{\nabla u_\Omega \cdot D^s u_\Omega}{\sqrt{1+|\nabla u_\Omega|^2}} = |D^s u_\Omega| \quad \text{in}\Omega, \tag{1.19}$$

$$\frac{(\nu \cdot \nabla u_\Omega) u_\Gamma}{\sqrt{1+|\nabla u_\Omega|^2}} = -|u_\Gamma| \quad \text{on } \Gamma. \tag{1.20}$$

These are the equations on the singular parts in Ω and on Γ.

Therefore, in the case of $D^s u_\Omega = 0$, the critical point $u \in X$ has the interior regularity that $u = u_\Omega$ belongs to $W^{1,1}(\Omega)$ and it satisfies (1.18). And the boundary condition (1.20) implies

$$u_\Gamma = 0 \quad \text{or} \quad (\nu \cdot \nabla u_\Omega) = \begin{cases} -\infty & \text{if } u_\Omega > 0, \\ +\infty & \text{if } u_\Omega < 0, \end{cases}$$

because $\frac{(\nu \cdot \nabla u_\Omega)}{\sqrt{1+|\nabla u_\Omega|^2}} = -\mathrm{sgn} u_\Gamma$ means $|\nu \cdot \nabla u_\Omega| = |\nabla u_\Omega| = +\infty$.

F. Demengel and R. Temam [3] have obtained the similar expression as in Theorem 1.2 for the subdifferntial related to the minimal surface operator. For the similar expression on the subdifferential related to 1-Laplace operator, see B. Kawohl and F. Schuricht [9] and F. Demengel [4].

2 The area functional and its convex conjugate

Let a^* be the convex conjugate of a in $\mathbb{R}^N$, namely,

$$a^*(p) = \sup\{p \cdot v - a(v) : v \in \mathbb{R}^N\} \quad \text{for } p \in \mathbb{R}^N.$$

Observe that

$$a^*(p) = \begin{cases} 1 - \sqrt{1-|p|^2} & \text{if } |p| \le 1, \\ +\infty & \text{otherwise.} \end{cases} \tag{2.21}$$

Let $BV_c(\mathscr{B})$ be all of functions in $BV(\mathscr{B})$ with compact support in $\mathscr{B}$.

Lemma 2.1. *For* $u \in BV_c(\mathscr{B})$,

$$\mathcal{A}(u) = \sup\{\int_{\mathscr{B}} u\mathrm{div}p\ dx - \int_{\mathscr{B}} a^*(p)dx : p \in V\}, \tag{2.22}$$

where

$$V = \{p \in L^1(\mathscr{B} : \mathbb{R}^N) : \mathrm{div}p \in L^N(\mathscr{B})\}. \tag{2.23}$$

Proof.

1.

$$\mathcal{A}(u) = \sup\{\int_{\mathscr{B}} u\mathrm{div}\eta\ dx - \int_{\mathscr{B}} a^*(\eta)dx : \eta \in C_c^1(\mathscr{B}; \mathbb{R}^N), |\eta| \le 1\}. \tag{2.24}$$

Indeed, the condition $|\hat{\eta}| \le 1$ in (1.5) implies

$$|\eta_0| \le \sqrt{1 - |\eta|^2}, \quad |\eta| \le 1.$$

Thus $\int_{\mathscr{B}} \eta_0 dx \le \int_{\mathscr{B}} \sqrt{1 - |\eta|^2} dx$ and the right-hand side of (1.5)(with $U = \mathscr{B}$) is dominated by that of (2.24). On the other hand, the continuous function $0 \le \sqrt{1 - |\eta|^2} \le 1$ on $\mathscr{B}$ can be approximated by C_0^1-function ξ with $0 \le \xi \le \sqrt{1 - |\eta|^2}$ in L^1 sense. Thus (2.24) holds.

2. Since $C_c^1(\mathscr{B}; \mathbb{R}^N) \subset V$,

$$\mathcal{A}(u) \le \sup\{\int_{\mathscr{B}} u\mathrm{div}p\ dx - \int_{\mathscr{B}} a^*(p)dx : p \in V\}. \tag{2.25}$$

3. By (2.21), $\int_{\mathscr{B}} a^*(p)dx = +\infty$ if $a^* \circ p \notin L^1(\mathscr{B})$. Therefor, V in the right-hand side of (2.25) can be replaced by V_1:

$$\begin{aligned} V_1 =& \{p \in L^1(\mathscr{B}; \mathbb{R}^N) : a^* \circ p \in L^1(\mathscr{B}), \mathrm{div}p \in L^N(\mathscr{B})\} \\ =& \{p \in L^\infty(\mathscr{B}; \mathbb{R}^N) : |p|_\infty \le 1, \mathrm{div}p \in L^N(\mathscr{B})\} \end{aligned} \tag{2.26}$$

Since u has a compact support $K \equiv \mathrm{supp}[u]$,

$$\int_{\mathscr{B}} u\mathrm{div}p\ dx = \int_K u\mathrm{div}p\ dx.$$

Let O_1, O_2 be open sets with $K \subset O_1 \subset\subset O_2 \subset\subset \mathscr{B}$ and ψ be C^∞-function with $0 \le \psi \le 1$, $\psi = 1$ on O_1 and $\psi = 0$ outside O_2. Then $\mathrm{div}(\psi p) = \mathrm{div}p$ on K and $|\psi p| \le |p|$. By the monotonicity of $a^*(p)$ with respect to $|p|$,

$$\begin{aligned} \mathcal{A}(u) =& \sup\{\int_{\mathscr{B}} u\mathrm{div}p\ dx - \int_{\mathscr{B}} a^*(p)dx : p \in V\} \\ =& \sup\{\int_{\mathscr{B}} u\mathrm{div}p\ dx - \int_{\mathscr{B}} a^*(p)dx : p \in V_1,\ \mathrm{supp}p \subset \mathscr{B}\}. \end{aligned}$$

4. To complete proof, it is sufficient to approximate the above integrals by ones whose p is replaced by functions $\eta \in C_c^1(\mathscr{B};\mathbb{R}^N)$ with $|\eta| \leq 1$. Let ρ_ϵ be a standard mollifier and set $p_\epsilon = \rho_\epsilon * p$. Then $p_\epsilon \in C_c^1(\mathscr{B};\mathbb{R}^N)$ with $|p_\epsilon| \leq 1$ for small $\epsilon > 0$ and

$$\begin{aligned} &p_{\epsilon_j} \to p \quad \text{a.e.}, \\ &\text{div} p_{\epsilon_j} = \rho_{\epsilon_j} * \text{div} p \to \text{div} p \quad \text{in } L^N(\mathscr{B}) \ , \end{aligned}$$

as $\epsilon_j \to 0$. Thus

$$\int_{\mathscr{B}} u \text{div} p_{\epsilon_j} dx \to \int_{\mathscr{B}} u \text{div} p \ dx,$$

and, by the Lebesgue's dominated convergence theorem,

$$\int_{\mathscr{B}} a^*(p_{\epsilon_j}) dx \to \int_{\mathscr{B}} a^*(p) dx.$$

□

Remark. Set

$$W = \{p \in L^\infty(\mathscr{B};\mathbb{R}^N) : \text{div} p \in L^N(\mathscr{B})\}.$$

Since $V_1 \subset W \subset V$, as we state in the proof, for $u \in BV_c(\mathscr{B})$ we have

$$\begin{aligned} \mathcal{A}(u) =& \sup\{\int_{\mathscr{B}} u \text{div} p \ dx - \int_{\mathscr{B}} a^*(p) dx : p \in V\} \\ =& \sup\{\int_{\mathscr{B}} u \text{div} p \ dx - \int_{\mathscr{B}} a^*(p) dx : p \in W\} \\ =& \sup\{\int_{\mathscr{B}} u \text{div} p \ dx - \int_{\mathscr{B}} a^*(p) dx : p \in V_1\} \end{aligned}$$

and we can impose p on its compact support in $\mathscr{B}$. Moreover, since a^* is even, the above integral $\int_{\mathscr{B}} u \text{div} p \ dx$ can be replaced by $\pm \int_{\mathscr{B}} u \text{div} p \ dx$.

Let U be a bounded domain in $\mathbb{R}^N$. For any $u \in BV(U)$ and any $p \in W(U) = \{p \in L^\infty(U;\mathbb{R}^N) : \text{div} p \in L^N(U)\}$, define the distribution $p \cdot Du$ by

$$\int_U \varphi(p \cdot Du) = -\int_U (\text{div} p) u \varphi dx - \int_U (p \cdot \nabla \varphi) u dx \tag{2.27}$$

for all $\varphi \in C_c^\infty(U)$. R. Kohn and R. Temam ([10] Proposition 1.1) have shown that $p \cdot Du$ is a bounded measure with $|p \cdot Du| \leq ||p||_\infty |Du|$ and obtained the Green's formula:

$$\int_{\partial U} (p \cdot \nu) u_{\partial U} \phi d\mathscr{H}^{N-1} = \int_U (p \cdot Du)\phi + \int_U (\text{div} p) u \phi dx + \int_U (p \cdot \nabla \phi) u dx \tag{2.28}$$

for all $\phi \in C^1(\overline{U})$.

For $u \in BV_c(\mathscr{B})$ and $p \in W$, define $p \cdot Du$ by (2.27) with $U = \mathscr{B}$. Choosing $\varphi \in C_c^\infty(\mathscr{B})$ with $\varphi = 1$ on supp$u$, we have

$$\int_{\mathscr{B}} p \cdot Du = -\int_{\mathscr{B}} (\text{div}p) u dx. \tag{2.29}$$

For $u \in X$, the restriction u_Ω of u on Ω belongs to $BV(\Omega)$. Defining $p \cdot Du_\Omega$ by (2.27)with $U = \Omega$ and $u = u_\Omega$, we have

$$\int_{\partial\Gamma} (p \cdot \nu) u_\Gamma \phi d\mathscr{H}^{N-1} = \int_\Omega (p \cdot Du_\Omega)\phi + \int_\Omega (\text{div}p) u_\Omega \phi dx + \int_\Omega (p \cdot \nabla\phi) u_\Omega dx. \tag{2.30}$$

Any $\varphi \in C_c^\infty(\mathscr{B})$ can be regard as $\varphi \in C^1(\overline{\Omega})$ and

$$\int_{\mathscr{B}} (\text{div}u)\varphi dx + \int_{\mathscr{B}} (p \cdot \nabla\varphi) u dx = \int_\Omega (\text{div}u_\Omega)\varphi dx + \int_\Omega (p \cdot \nabla\varphi) u_\Omega dx.$$

By (2.27)(with $U = \mathscr{B}$) , (2.30) and the above equation, we have

$$-\int_{\mathscr{B}} \varphi(p \cdot Du) = \int_\Gamma (p \cdot \nu) u_\Gamma \varphi d\mathscr{H}^{N-1} - \int_\Omega (p \cdot Du_\Omega)\varphi dx$$

for all $\varphi \in C_c^\infty(\mathscr{B})$. Thus,

$$p \cdot Du = p \cdot Du_\Omega - (p \cdot \nu) u_\Gamma d\mathscr{H}^{N-1} \tag{2.31}$$

for $u \in X$.
Remark. Choose $p = e_i$ $(1 \le i \le N)$ in the canonical basis $\{e_1, \cdots, e_N\}$ of $\mathbb{R}^N$. Then (2.31) yields (1.13).

Remark 2.2. By (2.29) and the remark given after the proof of Lemma 2.1,

$$\begin{aligned} \mathcal{A}(u) &= \sup\{\pm \int_{\mathscr{B}} u\text{div}p\ dx - \int_{\mathscr{B}} a^*(p) dx : p \in Y\} \\ &= \sup\{\mp \int_{\mathscr{B}} p \cdot Du - \int_{\mathscr{B}} a^*(p) dx : p \in Y\} \end{aligned} \tag{2.32}$$

for $u \in BV_c(\mathscr{B})$, where $Y = V_1$, W , $V_{1,c} = \{p \in V_1 : \text{supp}[p] \subset \mathscr{B}\}$ or $W_c = \{p \in W : \text{supp}[p] \subset \mathscr{B}\}$.

Define a functional $\mathcal{A}^*$ on W by

$$\mathcal{A}^*(p) = \int_{\mathscr{B}} a^*(p) dx \tag{2.33}$$

for $p \in W$. Then

$$\mathcal{A}^*(p) \begin{cases} \le |\mathscr{B}| & \text{if } p \in V_1, \\ = \infty & \text{otherwise.} \end{cases}$$

Since the right-hand side of (2.32) is determined ony by Du, we denote it by $\mathcal{B}(Du)$.

Definition 2.3. For $u \in BV_c(\mathscr{B})$, Define a functional $\mathcal{B}$ of the Radon measure Du by

$$\mathcal{B}(Du) = \sup\{\int_{\mathscr{B}} p \cdot Du - \mathcal{A}^*(p) : p \in V_1\}. \tag{2.34}$$

Remark 2.4. By (2.32), of course,

$$\begin{aligned}\mathcal{B}(Du) =& \mathcal{A}(u) \\ =& \sup\{-\int_{\mathscr{B}} u\text{div}p\ dx - \mathcal{A}^*(p) : p \in Y\} \\ =& \sup\{\int_{\mathscr{B}} p \cdot Du - \mathcal{A}^*(p) : p \in Y\}\end{aligned}$$

where $Y = V_1, V_{1,c}, W$ or W_c.

For a Radon measure $\mu = \mu^a + \mu^s$, where $\mu^a = f(x)dx$, μ^s be respectively the absolute continuous and singular parts with respect to the Lebesgue measure, define a measure $a(\mu)$ by $a(\mu) = a(f(x))dx + a_\infty(\mu^s)$, where $a_\infty(x) = \lim_{t\to\infty} a(tx)/t = |x|$, i.e.,

$$a(\mu) = a(f(x))dx + |\mu^s|.$$

F. Demengel and R. Temam [3] have shown

$$\int_{\mathscr{B}} \varphi a(\mu) = \sup\{\int_{\mathscr{B}} \varphi p \cdot d\mu - \int_{\mathscr{B}} \varphi a^*(p)dx : p \in V_1\}$$

for all $\varphi \in C_c^\infty(\mathscr{B})$ with $\varphi \geq 0$. Let $u \in BV_c(\mathscr{B})$. Put $\mu = Du = \nabla u dx + D^s u$. Choosing φ with $\varphi = 1$ on supp$[u]$, we have

$$\int_{\mathscr{B}} a(Du) = \sup\{\int_{\mathscr{B}} p \cdot Du - \int_{\mathscr{B}} a^*(p)dx : p \in V_1\}.$$

Thus

$$\begin{aligned}\int_{\mathscr{B}} a(Du) =& \int_{\mathscr{B}} a(\nabla u)dx + |D^s u|(\mathscr{B}) \\ =& \mathcal{B}(Du) \\ =& \mathcal{A}(u)\end{aligned} \tag{2.35}$$

for $u \in BV_c(\mathscr{B})$. This shows that the equation (1.7) holds for the measure $a(Du)$ defined above.

Denote by $[Du]_{\overline{\Omega}}$ the restriction of Du on $\overline{\Omega}$. By the monotonicity of $a(v)$ with respect to $|v|$ and the above equation,

$$\mathcal{B}([Du]_{\overline{\Omega}}) \leq \mathcal{B}(Du).$$

For $h \in W^{1,1}(\mathscr{B})$ with compact support, we write simply $\mathcal{B}(\nabla h) = \mathcal{B}(\nabla h dx)$, i.e.,

$$\mathcal{B}(\nabla h) = \int_{\mathscr{B}} a(\nabla h) dx.$$

Lemma 2.5. *Let $p \in W$, then*

$$\int_{\overline{\Omega}} a^*(p) dx = \sup\{-\int_{\mathscr{B}} u \text{div} p \; dx - \mathcal{B}(Du) : u \in X\} \tag{2.36}$$

Remark. (2.36) hold even if the left-hand side equals to $+\infty$.

Proof of Lemma 2.5.

1. Fix any $u \in X$. Let $\varphi \in C_0^\infty(\mathscr{B})$ satisfying $0 \leq \varphi \leq 1$ and $\varphi = 1$ on $\overline{\Omega}$, then $\varphi p \in W_c$ for all $p \in W$. Since $\text{supp}[Du] \subset \overline{\Omega}$, by the definition 2.3 and Remark 2.4, we have

$$\mathcal{B}(Du) \geq \int_{\overline{\Omega}} p \cdot Du - \int_{\mathscr{B}} a^*(\varphi p) dx,$$

for all $p \in W$. Letting $\varphi \searrow \chi_{\overline{\Omega}}$ yields

$$\mathcal{B}(Du) \geq \int_{\overline{\Omega}} p \cdot Du - \int_{\overline{\Omega}} a^*(p) dx,$$

Thus,

$$\begin{aligned} \int_{\overline{\Omega}} a^*(p) dx \geq& \sup\{\int_{\mathscr{B}} p \cdot Du - \mathcal{B}(Du) : u \in X\} \\ =& \sup\{-\int_{\mathscr{B}} u \text{div} p \; dx - \mathcal{B}(Du) : u \in X\} \end{aligned} \tag{2.37}$$

for all $p \in W$.

2. Define a functional A on $L^1(\mathscr{B}; \mathbb{R}^N)$ by

$$A(g) = \int_{\mathscr{B}} a(g) dx \quad \text{for } g \in L^1(\mathscr{B}; \mathbb{R}^N).$$

I. Eckeland and R. Temam have shown that the convex conjugate A^* of A on the dual space $L^\infty(\mathscr{B}; \mathbb{R}^N)$ is given by $A^*(\cdot) = \int_{\mathscr{B}} a^*(\cdot) dx$ (see Lemma1.1. Chap. V [6]), namely,

$$\begin{aligned} \int_{\mathscr{B}} a^*(q) dx =& A^*(q) \\ =& \sup\{\int_{\mathscr{B}} q \cdot g dx - \int_{\mathscr{B}} a(g) dx : g \in L^1(\mathscr{B}; \mathbb{R}^N)\} \end{aligned}$$

for all $q \in L^\infty(\mathscr{B};\mathbb{R}^N)$. Set $q = \chi_{\overline{\Omega}} p$, then

$$\int_{\overline{\Omega}} a^*(p)dx = \sup\{\int_{\overline{\Omega}} p \cdot g dx - \int_{\mathscr{B}} a(g)dx : g \in L^1(\mathscr{B};\mathbb{R}^N)\}$$

Note that, for $g \in L^1(\mathscr{B};\mathbb{R}^N)$, $\chi_{\overline{\Omega}} g \in L^1(\mathscr{B};\mathbb{R}^N)$ and $\int_{\mathscr{B}} a(g)dx \geq \int_{\mathscr{B}} a(\chi_{\overline{\Omega}} g)dx = \int_{\overline{\Omega}} a(g)dx$. Thus

$$\begin{aligned}
\int_{\overline{\Omega}} a^*(p)dx &= \sup\{\int_{\overline{\Omega}} p \cdot g dx - \int_{\overline{\Omega}} a(g)dx : g \in L^1(\mathscr{B};\mathbb{R}^N)\} \\
&\leq \sup\{\int_{\overline{\Omega}} p \cdot Du dx - \int_{\overline{\Omega}} a(Du) : u \in BV(\mathscr{B})\} \\
&= \sup\{\int_{\mathscr{B}} p \cdot Du dx - \int_{\mathscr{B}} a(Du) : u \in X\} \\
&= \sup\{-\int_{\mathscr{B}} u \mathrm{div} p \ dx - \int_{\mathscr{B}} a(Du) : u \in X\}. \qquad (2.38)
\end{aligned}$$

(2.37) and (2.38) show the lemma.

□

Since $\mathcal{B}(Du) = \mathcal{A}(u)$, by Lemma 2.5,

$$\int_{\overline{\Omega}} a^*(p)dx = \sup\{-\int_{\mathscr{B}} u \mathrm{div} p \ dx - \mathcal{A}(u) : u \in X\}$$

for $p \in W$. The right-hand side is depending only on $\pm\mathrm{div} p \in L^N(\mathscr{B})$. Set

$$Z = \{z \in L^N(\mathscr{B}) : z = -\mathrm{div} p,\ p \in W_c\}, \qquad (2.39)$$

and define a functional $\underline{\mathcal{A}^*}$ on $L^N(\mathscr{B})$ by

$$\underline{\mathcal{A}^*}(z) = \begin{cases} \sup\{-\int_{\mathscr{B}} uz dx - \mathcal{A}(u) : u \in X\} & \text{if } z \in Z \\ +\infty & \text{otherwise} \end{cases}. \qquad (2.40)$$

Note that $\underline{\mathcal{A}^*}(z) = \mathcal{A}^*(-p) = \mathcal{A}^*(p) = \underline{\mathcal{A}^*}(-z)$.

Lemma 2.6. *$\underline{\mathcal{A}^*}$ is lower semicontinuous with respect to L^N-topology.*

Proof It is sufficient to show that the set $\{z \in L^N(\mathscr{B}) : \underline{\mathcal{A}^*}(z) \leq \alpha\}$ is closed for any $\alpha \in \mathbb{R}$. Suppose the sequence $\{z_n\}$ satisfies $\underline{\mathcal{A}^*}(z_n) \leq \alpha$ and $z_n \to z$ in $L^N(\mathscr{B})$. Thus, $z_n = -\mathrm{div} p_n$ and $|p_n|_\infty \leq 1$ and we may assume $\mathrm{supp}[p_n] \subset O$ for an open set O with $\overline{\Omega} \subset O \subset\subset \mathscr{B}$. There is a subsequence $\{p_{n'}\}$ and $p \in L^\infty(\mathscr{B};\mathbb{R}^N)$ such that

$$p_{n'} \to p \quad \text{weakly}^*\ ,$$

thus $|p|_\infty \leq 1$ and $\text{supp}[p] \subset O$. Since

$$\int_{\mathscr{B}} z_{n'} \varphi dx = -\int_{\mathscr{B}} (\text{div} p_{n'}) \varphi dx = \int_{\mathscr{B}} p_{n'} \cdot \nabla \varphi dx$$
$$\to \int_{\mathscr{B}} p \cdot \nabla \varphi dx = -\int_{\mathscr{B}} (\text{div} p) \varphi dx,$$

and

$$\int_{\mathscr{B}} z_{n'} \varphi dx \to \int_{\mathscr{B}} z \varphi dx$$

for all $\varphi \in C_c^\infty(\mathscr{B})$. Thus $z = -\text{div} p$. Since $z_{n'} \to z$ and

$$\int_{\mathscr{B}} u z_{n'} dx - \mathcal{A}(u) \leq \alpha,$$

$$\int_{\mathscr{B}} u z dx - \mathcal{A}(u) = \underline{\mathcal{A}^*}(z) \leq \alpha.$$

This shows the lemma.

□

Note that, for any $z \in Z$, if $\underline{\mathcal{A}^*}(z)$ is finite then $z = -\text{div} p$ with $p \in W_c$ and

$$\underline{\mathcal{A}^*}(z) = \int_{\Omega} a^*(p) dx. \tag{2.41}$$

3 Proof of Theorem 1.2

Proposition 3.1. *Let $u \in X$ and suppose that there exists $z \in L^N(\mathscr{B})$ holding the inequality,*

$$\mathcal{A}(v) - \mathcal{A}(u) \geq \lambda \int_{\mathscr{B}} z(v-u) dx \quad \textit{for all} \;\; v \in X. \tag{3.42}$$

Then there exists $p \in V_1$ with $-\text{div} p = z$ such that

$$p = \frac{\nabla u_\Omega}{\sqrt{1 + |\nabla u_\Omega|^2}} \quad \textit{in } \Omega, \tag{3.43}$$
$$p \cdot D^s u_\Omega = |D^s u_\Omega| \quad \textit{in } \Omega, \tag{3.44}$$
$$(\nu \cdot p) u_\Gamma = -|u_\Gamma| \quad \textit{on } \Gamma. \tag{3.45}$$

Proof.

1. We may assume that z in (3.42) has a compact support in $\mathscr{B}$. Indeed, if not, by applying Lemma 4.1 in Appendix, we have $p \in W_c$ satisfying $z = -\text{div} p$

on $\overline{\Omega}$. Thus we can replace z in the inequality by $\hat{z} \equiv -\text{div}p$ which has a compact support.

2. Thus there exists $z = -\text{div}p$ with p in W_c such that

$$\int_{\mathscr{B}} zudx - \mathcal{A}(u) \geq \int zvdx - \mathcal{A}(v) \quad \text{for all } v \in X.$$

Since $u \in X$, this implies

$$\begin{aligned}\int_{\mathscr{B}} zudx - \mathcal{A}(u) =& \sup\{\int_{\mathscr{B}} zvdx - \mathcal{A}(v) : v \in X\} \\ =& \sup\{-\int_{\mathscr{B}} v\text{div}p\ dx - \mathcal{A}(v) : v \in X\} \\ =& \mathcal{A}^*(p) = \underline{\mathcal{A}^*}(z).\end{aligned}$$

The left-hand side is finite and so is $\mathcal{A}^*(p)$. Hence, $p \in V_{1,c}$ and by (2.41)

$$\begin{aligned}-\int_{\mathscr{B}} u\text{div}p\ dx - \mathcal{A}(u) &= \int_{\overline{\Omega}} a^*(p)dx, \\ \int_{\mathscr{B}} p \cdot Dudx - \int_{\mathscr{B}} a(Du) &= \int_{\overline{\Omega}} a^*(p)dx.\end{aligned}$$

Note that the domain $\mathscr{B}$ of integration in the left-hand side can be regard as $\overline{\Omega}$ because the support of Du is in it. Since $Du = Du_\Omega - u_\Gamma \nu d\mathscr{H}^{n-1}$ and $Du_\Omega = \nabla u_\Omega dx + D^s u_\Omega$, we have

$$\begin{aligned}&\int_\Omega p \cdot \nabla u_\Omega dx + \int_\Omega p \cdot D^s u_\Omega - \int_\Gamma (p \cdot \nu) u_\Gamma d\mathscr{H}^{N-1} \\ &- \int_\Omega a(\nabla u_\Omega)dx - \int_\Omega |D^s u_\Omega| - \int_\Gamma |u_\Gamma| d\mathscr{H}^{N-1} \\ &= \int_\Omega a^*(p)dx, \\ 0 =& \int_\Omega (a^*(p) - ((p \cdot \nabla u_\Omega) - a(\nabla u_\Omega)))dx \\ &+ \int_\Omega (|D^s u_\Omega| - (p \cdot D^s u_\Omega)) \\ &+ \int_\Gamma (|u_\Gamma| + (p \cdot \nu) u_\Gamma) d\mathscr{H}^{N-1}.\end{aligned}$$

Note that each integrand of the three terms of the last equation is nonnegative. Hence the equation implies the integrands vanish identically.

$$\begin{aligned}a^*(p) =& p \cdot \nabla u_\Omega - a(\nabla u_\Omega) \\ p \cdot D^s u_\Omega =& |D^s u_\Omega| \\ (p \cdot \nu) u_\Gamma =& -|u_\Gamma|\end{aligned} \tag{3.46}$$

(3.46) implies $p = a'(\nabla u_\Omega)$ which is equal to (3.43). Other two equations are (3.44) and (3.45).

□

Proof of Theorem 1.2

Since $u \in X$ is a solution of variational inequality (1.11), setting $z = g(x,u)$ and noting $\text{supp}[g(x,u)] \subset \overline{\Omega}$, apply Proposition 3.1.

□

Remark 3.2. Finally, we illustrate an analogy between our approach developed here and the process in the theoretical mechanics deriving the Hamilton's canonical form from the Euler-Lagrange equation. In what follows, we often describe formally without the mathematical accuracy.

Let regard our functional $\mathcal{L}_\lambda$ as a Lagrangean of the form

$$\mathcal{L}_\lambda(u, Du) = \mathcal{B}(Du) - \lambda\mathcal{G}(u). \tag{3.47}$$

In the case of a smooth u, its Euler-Lagrange equation is given by the second order differential equation in (0.1). For the functional $\mathcal{L}_\lambda$ involving the Radon measure Du, we cannot derive directly such an equation. By using the Legendre transformation

$$\mathcal{A}^*(p) = \sup\{\int_{\mathscr{B}} p \cdot Du dx - \mathcal{B}(Du) : u \in X\}, \tag{3.48}$$

the equation in which $\mathcal{A}^*(p)$ replaced $\int_{\overline{\Omega}} a^*(p)dx$ is given in Lemma 2.5, and setting the Hamiltonian

$$\mathcal{H}(u,p) = \mathcal{A}^*(p) - \lambda\mathcal{G}(u), \tag{3.49}$$

we introduce a system of the first oder differential equations

$$\begin{cases} Du = -\dfrac{\partial\mathcal{H}}{\partial p} = -\dfrac{\partial\mathcal{A}^*}{\partial p}, \\ D \cdot p = \dfrac{\partial\mathcal{H}}{\partial u} = -\lambda\dfrac{\partial\mathcal{G}}{\partial u} \end{cases}. \tag{3.50}$$

In the theoretical dynamics, the canonical form governing the motion of a mass has the same type as (3.50) where u and $p = \mathcal{B}'(Du)$ represent the position and the momentum of the mass respectively and $D = \frac{d}{dt}$. As we stated in the above, (3.48) is written formally

$$\underline{\mathcal{A}^*}(z) = \sup\{< z, u > -\mathcal{A}(u) : u \in X\}$$

where $\mathcal{A}(u) = \mathcal{B}(Du)$, $z = -\text{div}p$. The variational inequality (1.11) implies

$$\lambda\int_{\mathscr{B}} gudx - \mathcal{A}(u) = \underline{\mathcal{A}^*}(\lambda g) = \mathcal{A}^*(p)$$

with $\lambda g = -\mathrm{div}p$. By Lemma 2.5(the key lemma),

$$\begin{aligned}\int_{\overline{\Omega}} a^*(p)dx &= \int_{\mathscr{B}} (-\mathrm{div}p)udx - \mathcal{A}(u)\\ &= \int_{\mathscr{B}} (-\mathrm{div}p)udx - \int_{\mathscr{B}} a(Du))\end{aligned}$$

By using $Du = \nabla u_\Omega dx + D^s u_\Omega - u_\Gamma \nu d\mathscr{H}^{n-1}$,

$$\begin{aligned}&\int_{\mathscr{B}} (a^*(p) - p\cdot\nabla u_\Omega - a(\nabla u_\Omega))dx\\ &+\int_{\mathscr{B}} (|D^s u| - p\cdot D^s u)\\ &+\int_{\Gamma} (|u_\Gamma| + (p\cdot\nu))d\mathscr{H}^{N-1}\\ &=0.\end{aligned}$$

By this, instead of the formal relation $p = \mathcal{B}'(Du)$ between the momentum and the velocity, we obtained more explicit relations (3.43)-(3.45) in Proposition 3.1.

Since $\mathcal{A}^*(p) = \int_{\mathscr{B}} a^*(p)dx$ and $\mathcal{G}(u) = \int_{\mathscr{B}} G(x,u)dx$,

$$\begin{aligned}&< \frac{\partial \mathcal{A}^*}{\partial p}, \phi > = \int_{\mathcal{B}} (a^*)'(p)\cdot\phi dx = -\int_{\mathscr{B}} \frac{p\cdot\phi}{\sqrt{1-|p|^2}}dx,\\ &< \frac{\partial \mathcal{G}}{\partial u}, \varphi > = \int_{\mathscr{B}} g(x,u)\varphi dx,\end{aligned}$$

we can write the equations in (3.50) in weak form:

$$\begin{aligned}&< u, -\mathrm{div}\phi > = < \frac{p}{\sqrt{1-|p|^2}}, \phi > \quad \text{for all} \quad \phi\in C_c^\infty(\mathscr{B};\mathbb{R}^N),\\ &< p, \nabla\varphi > = -\lambda < g, \varphi > \qquad \text{for all} \quad \varphi\in C_c^\infty(\mathscr{B};\mathbb{R}).\end{aligned}$$

Hence

$$Du = \nabla u_\Omega = \frac{p}{\sqrt{1-|p|^2}},$$

which is equivalent (3.43), and

$$\mathrm{div}p = -\lambda g(x,u).$$

4 Appendix.

Lemma 4.1. *For any $z \in L^N(\mathscr{B})$, there exists $\hat{z} \in L^N(\mathscr{B})$ with compact support such that $z = \hat{z}$ on $\overline{\Omega}$ and $\hat{z} = -\mathrm{div}p$ for some $p \in L^\infty(\mathscr{B};\mathbb{R}^N)$ with compact support in $\mathscr{B}$.*

Proof.

Let O_1, O_2 be open sets satisfying $\overline{\Omega} \subset O_1 \subset\subset O_2 \subset\subset \mathscr{B}$, and ψ be a C_c^∞-funftion such that $0 \le \psi \le 1$, $\psi = 1$ on $\overline{O_1}$ and $\psi = 0$ outside O_2. Let l_i be a line parallel to x_i-axis, say, $l_i = \{(x_1, \cdots, x_{i-1}, \xi, x_{i+1}, \cdots, x_N) : \xi \in \mathbb{R}\}$. For each $y_i = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_N) \in \mathbb{R}^{N-1}$, we write simply $l_i = \{(\xi, y_i) : \xi \in \mathbb{R}\}$. The line segment $l_i \cap \mathscr{B}$ can be expressed by l_i with $|\xi| < R_i$ for some $R_i = R_i(y_i) > 0$. Set $z_0 = \frac{1}{N}\psi z$ and

$$w_i(x) = -\int_{-R_i}^{x_i} z_0(\xi, y_i) d\xi$$

for $|x_i| \le R_i$ and $1 \le i \le N$. Then

$$w_i \in L^\infty(\mathscr{B}), \quad \frac{\partial w_i}{\partial x_i} = -z_0.$$

Denote $w = (w_1, \cdots, w_N)$ and set $p = \psi w$, then p has a support in $\overline{O_2}$, belongs to $L^\infty(\mathscr{B}; \mathbb{R}^N)$ and is equal to w in $\overline{O_1}$. Hence, $\hat{z} = -\text{div} p$ has a compact support in $\overline{O_2}$ and

$$\hat{z} = -\text{div} p = N z_0 = z \quad \text{in } O_1$$

□

References

[1] L. Ambrosio, N. Fusco and D. Pallara, Functions of Bounded Variation and Free Discontinuity Problems, Oxford Mathematical Monographs, Oxford University Press, New York, 2000.

[2] L. C. Evans and R. F. Gariepy, Measure Theory and Fine Properties of Functions, Studies in Advanced Mathematics, CRC press, 1992.

[3] F. Demengel and R. Temam, Convex functions of a measure and applications, Indiana Univ. Math. J. Vol.33, No.5(1984), 673-709

[4] F. Demengel, On the stability under perturbation of solutions of non-linear problems on $W^{1,1}(\Omega)$, Asymptotic Anal. 21 (1999), 209-220

[5] M. Degiovanni and M. Marzocchi, A critical point theory for nonsmooth functionals, Ann. Math. Pure Appl. 167 (1994), 73-100

[6] I. Ekeland and R. Témam, Convex Analysis and Variational Problems, SIAM Classics in Applied Mathematics 28

[7] E. Giusti, Minimal Surfaces and Functions of Bounded Variations, Birkhäuser, Basel, 1984

[8] G. Katriel, Mountain pass theorems and global homeomorphism theorems, Ann. Inst. H. Poincaré Anal. Non Linéare 11 (1994), 189-209

[9] B. Kawohl and F. Schuricht, Dirichlet problems for the 1-Laplace operator, including the eigenvalue problem, Commun. Contemp. Math. 9(4) (2007), 515-544

[10] R. Kohn and R. Temam, Dual spaces of stresses and strains, with applications to Hencky plasticity, Appl. Math. Optim. 10 (1983), 1-35

[11] V.K. Le, Some existence results on nontrivial solutions of the prescribed mean curvature equation, Adv. Nonlinear Studi. 5 (2005), 133-161

[12] V.K. Le, A Ljusternik-Schnirelmann type theorem for eigenvalues of a prescribed mean curvature problem, Nonlinear Anal. 64 (2006), 1503-1527