Prescribed Mean Curvature Equations
for
Functions of Bounded Variation

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Abstract

Let \( L(u) = L(u, \nabla u) \) be a functional on \( W^{1,1}(\Omega) \) whose formal Euler-Lagrange equation at the critical point \( u \) of \( L \) is the prescribed mean curvature equation:

\[
-\text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = g(x, u).
\]

Suppose \( L(u) = L(u, Du) \) is a relaxed functional of \( L(u) \), the weakly lower semicontinuous extension of \( L \) on the space of functions of bounded variation. How does the relaxation affect the prescribed mean curvature equation? Instead of an Euler-Lagrange equation, we obtain here the so-called Euler-Lagrange system of equations which the critical points \( u \) of \( L \) and their derivatives \( Du \) necessarily satisfy.

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Introduction

We are concerned here with a Dirichlet boundary value problem (DBVP) of the prescribed mean curvature equation:

\[
\begin{aligned}
-\text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) &= \lambda g(x, u) & \text{in } \Omega, \\
\quad u &= 0 & \text{on } \Gamma.
\end{aligned}
\]

(0.1)

Here \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) \((N \geq 2)\) with sufficiently smooth boundary \( \Gamma = \partial \Omega \), \( g \) is a Carathéodory function and \( \lambda \) a positive real number. The
problem is derived (at least formally) as an Euler-Lagrange equation from a variational problem for the functional

\[ L_\lambda(u) = \int_\Omega a(\nabla u)dx - \lambda \int_\Omega G(x,u)dx \]  

(0.2)
on \text{W}^{1,1}(\Omega)$, where $a(v) = \sqrt{1+|v|^2} - 1$ on $v \in \mathbb{R}^N$ and $G(x,u) = \int_0^u g(x,s)ds$. However, the principal part $\mathcal{A}_0(u) = \int_\Omega a(\nabla u)dx$, called an area functional, is not lower semicontinuous with respect to the weak topology (or even in $L_1$-topology) of the space, and neither is $L_\lambda$. A standard approach to the variational problem of such a functional is to relax the functional in such a way the resulting functional becomes lower semicontinuous and to seek critical points of the relaxed functional. In the case of $L_\lambda$, the relaxed functional $L_\lambda$ is defined on $BV(\Omega)$, the space of functions of bounded variation on $\Omega$. The existence of a local minimum point and a non-minimal critical point of $L_\lambda$ has been proven by V. K. Le [11, 12].

We here direct our attention to the existence problem of the Euler-Lagrange equation itself like (0.1) for $L_\lambda$. How is it expressed if it exists? We cannot use the minimality, of course, to characterize the non-minimal critical points of $L_\lambda$. For the non-differentiable functional $L_\lambda$, its critical points cannot be defined as zeros of its derivative, and should be done in some indirect way. V. K. Le defines the critical point of $L_\lambda$ as a solution of a variational inequality. Therefore it is of interest to obtain some Euler-Lagrange equations like (0.1) which critical points necessarily satisfy. Since the critical point $u$ belongs to $BV(\Omega)$, its distributional derivative $Du$ is a bounded measure. Thus the expected Euler-Lagrange equation will involve the measure $Du$ together with $u$ as unknowns. The measure $Du$ can be divided into some parts which are singular each other. For instance, $Du = D^a u + D^s u$, where $D^a u$ (or $D^s u$) is the absolutely continuous (or singular respectively) part with respect to the $N$-dimensional Lebesgue measure $dx = d\mathcal{L}^N$. And the Euler-Lagrange equation may be described as a system of some equations, each of those governs one part of $Du$. In that case, we call it the Euler-Lagrange system and refer to it as the prescribed mean curvature equations of (DBVP) since the system itself governs a function $u \in BV(\Omega)$ regarded as a solution of (DBVP). For the critical point $u$ of $L_\lambda$ defined as a solution of the variational inequality proposed by Le, we obtain the prescribed mean curvature equations of (DBVP). To state our result more precisely, we give some preliminaries and a short summary of results gotten by V. K. Le.

1 Preliminaries and the main result

In this section, we give some preliminaries together with a short summary of the existence results of the critical points of $L_\lambda$ proved by Le. And then we state the main theorem of the present paper.
The space of functions of bounded variation on $\Omega$ is defined by
\[
BV(\Omega) = \{ u \in L^1(\Omega) : |Du|_{\Omega} < +\infty \} \tag{1.3}
\]
where
\[
|Du|_{\Omega} = \sup\left\{ \int_{\Omega} u \text{div} \eta \, dx : \eta = (\eta_1, \ldots, \eta_N) \in C^1_c(\Omega; \mathbb{R}^N), |\eta| \leq 1 \right\}
\tag{1.4}
\]
is called the total variation of $u$ on $\Omega$. $Du$ is the distributional derivative of $u$, namely,
\[
<Du, \eta> = -\int_{\Omega} u \text{div} \eta \, dx \quad \text{for all } \eta \in C^\infty_c(\Omega; \mathbb{R}^N),
\]
and it is a $\mathbb{R}^N$-valued Radon measure on $\Omega$ if $u \in BV(\Omega)$. $|Du|$ is a positive Radon measure satisfying
\[
|Du|(O) = \int_O |Du| = |Du|_O
\]
for all open set $O \subset \Omega$. $BV(\Omega)$ is a Banach space with the norm $\|u\| = ||u||_{L^1(\Omega)} + |Du|_{\Omega}$. It is well known that $BV(\Omega)$ is embedded in $L^1(\Omega)$ ($1^* = \frac{N}{N-1}$) continuously. The function $u \in BV(\Omega)$ has a trace $u_{\Gamma}$ which is the boundary value of $u$ on $\Gamma$ if $u \in C^1(\bar{\Omega})$. The mapping $\gamma : BV(\Omega) \to L^1(\Gamma)$ with $\gamma(u) = u_{\Gamma}$ is continuous.

The definition of the area functional $A(u : \Omega) = \int_{\Omega} a(Du)$ has been introduced by E. Giusti [7].

**Definition 1.1.** Let $U$ be a bounded domain in $\mathbb{R}^N$. Define an area functional $A$ on $L^1(\Omega)$ by
\[
A(u : U) = \sup\left\{ \int_U (\eta_0 + u \text{div} \eta - 1) \, d\tilde{\eta} : \tilde{\eta} = (\eta_0, \eta) \in C^1_c(U; \mathbb{R} \times \mathbb{R}^N), |\tilde{\eta}| \leq 1 \right\}
\tag{1.5}
\]
where $\eta = (\eta_1, \ldots, \eta_N)$ and $\text{div} \eta = \sum_{i=1}^N \frac{\partial \eta_i}{\partial x_i}$.

Observe that $A = A(\cdot : U)$ is a convex functional on $L^1(\Omega)$ with the effective domain $\text{dom} A = BV(U)$, i.e., $A(u : U)$ is finite if and only if $u \in BV(U)$, and is lower semicontinuous in $L^1$ and thus in $L^1$-toplogy.

The relaxation $A_1$ of $A_0$ on $W^{1,1}_0(\Omega)$ in the introduction, $A_0(u) = \int_{\Omega} a(\nabla u) \, dx$ for $u \in W^{1,1}_0(\Omega)$, is known as the functional on $BV(\Omega)$ defined by
\[
A_1(u) = A(u : \Omega) + \int_{\Gamma} |u_{\Gamma}| \, d\mathcal{H}^{N-1} \quad \text{for } u \in BV(\Omega) \tag{1.6}
\]
The integral in the right-hand side, which is the relaxation term for the boundary condition $u = 0$ on $\Gamma$, can be eliminated by extending function $u$ by taking
the value zero outside $\Omega$. Let $B$ be an open ball (with center $0$) such that $\overline{\Omega} \subset B$ and put

$$\bar{u} = \begin{cases} u & \text{in } \Omega \\ 0 & \text{in } B \setminus \Omega \end{cases},$$

then $\bar{u} \in BV(B)$ and

$$A_1(u) = A(\bar{u} : B).$$

Setting

$$X = \{u \in BV(B) : u = 0 \text{ on } B \setminus \Omega\},$$

we can replace (1.6) by

$$A(u : B) = \int_B a(Du) \quad (1.7)$$

for $u \in X$. Remark that the above integral is here a formal and convenient expression of the functional $A(u : B)$ defined in Definition 1.1. We give later another definition of the integral and prove that the above equality holds for $u \in BV(B)$ with compact support (see (2.35)). Since we only deal with functions in $X$ or $BV(B)$ in what follows, we write simply $A(\cdot : B)$.

We assume the following conditions:

(A.1) $g : B \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying

$$g(x, \xi) = 0 \quad \text{for } x \in B \setminus \Omega.$$

(A.2) There exists $q \in (1, 1^*)$ such that

$$|g(x, \xi)| \leq d_1|\xi|^{q-1} + d_2 \quad \text{for a.e. } x \in B, \text{ all } \xi \in \mathbb{R}$$

with some constant $d_1, d_2 > 0$.

Put

$$\mathcal{G}(u) = \int_B G(x, u)dx \quad (1.8)$$

for $u \in L^{1^*}(B)$. By (A.2), $\mathcal{G}$ is Fréchet differentiable and

$$\langle \mathcal{G}'(u), v \rangle = \int_B g(x, u)vdx \quad (1.9)$$

for $v \in L^{1^*}(B)$. The relaxation $L_\lambda$ of $L_\lambda$ in the introduction is a functional on $X$ denoted by

$$L_\lambda(u) = A(u) - \lambda \mathcal{G}(u) \quad (1.10)$$

for $u \in X$.

Under (A.1), (A.2) and some additional conditions, the existence of a local minimum point and a non-minimal critical point of functional $L_\lambda$ has been proven by V. K. Le [11]. Since $A$ is not differentiable, introducing the “weak slope” (in [5]) instead of the derivative and using the mountain pass
argument (in [8]), he proved the existence of the non-minimal critical point of $\mathcal{L}_\lambda$ for small $\lambda$, which is, as a result, a solution $u \in X$ of the variational inequality:

$$\mathcal{A}(v) - \mathcal{A}(u) - \lambda \int_B g(x, u)(v - u)dx \geq 0 \quad \text{for all } v \in X. \tag{1.11}$$

The minimum point $u$ of $\mathcal{L}_\lambda$ also satisfies (1.11). When a pair $(\lambda, u)$ satisfies (1.11) $\lambda$ is called an eigenvalue and $u$ its eigenfunction. By using a Ljusternik-Schnirelmann theory for (1.11), Le [12] also obtained infinite sequence of eigenvalues and eigenfunctions.

Since the functional $\mathcal{A}$ is convex on $X$, the inequality (1.11) implies

$$0 \in \partial \mathcal{A}(u) - \lambda \mathcal{G}'(u) \tag{1.12}$$

where $\partial \mathcal{A}(u)$ is the subdifferential of $\mathcal{A}$ at $u$. Thus (1.12) or the inequality (1.11) itself can be regard as a weak form of the Euler-Lagrange equation for the critical point of $\mathcal{L}_\lambda$. Our aim is to obtain a more explicit expression.

For $u \in X$, we denote by $u_\Omega$ the restriction of $u$ onto $\Omega$, then $u_\Omega \in BV(\Omega)$ and

$$Du = Du_\Omega - u_\Gamma \nu d\mathcal{H}^{N-1}, \tag{1.13}$$

where $u_\Gamma$ is the trace of $u_\Omega$ on $\Gamma$, $u_\Gamma = (u_\Omega)_\Gamma$ (see e.g. [1, 2]).

Let $U \subset \mathbb{R}^N$ be open and $v \in BV(U)$, $D^a v$ and $D^s v$ be respectively the absolutely continuous and singular parts of the Radon measure $Dv$. Denote the density $\frac{D^s v}{\mathcal{H}^{N-1}}$ by $\nabla v \in L^1(U)$, then

$$Dv = D^a v + D^s v = \nabla v dx + D^s v.$$ 

For $u \in X$, by (1.13), we have

$$D^a u = D^a u_\Omega = \nabla u_\Omega dx, \quad D^s u = D^s u_\Omega + u_\Gamma \nu d\mathcal{H}^{N-1}, \quad Du = \nabla u_\Omega dx + D^s u_\Omega - u_\Gamma \nu d\mathcal{H}^{N-1}.$$ 

The main result of the present paper ia as follows.

**Theorem 1.2.** Let $u \in X$ be a solution of the variational inequality (1.11), then there exists $p \in L^\infty(B; \mathbb{R}^N)$ such that

$$-\text{div} p = \lambda g(x, u_\Omega) \quad \text{in } \Omega, \tag{1.14}$$

$$p = \frac{\nabla u_\Omega}{\sqrt{1 + |\nabla u_\Omega|^2}} \quad \text{in } \Omega, \tag{1.15}$$

$$p \cdot D^s u_\Omega = |D^s u_\Omega| \quad \text{in } \Omega, \tag{1.16}$$

$$(\nu \cdot p) u_\Gamma = -|u_\Gamma| \quad \text{on } \Gamma. \tag{1.17}$$
where $\nu \cdot p$ is the weakly-defined trace of the normal component of $p$, which lies in $L^\infty(\Gamma)$.

**Remark 1.3.** In the above theorem, we can eliminate $p$. By (1.15), $p \in L^\infty$ is obvious. Substituting (1.15) into (1.14), we have the prescribed mean curvature equation on $u_\Omega$:

$$-\text{div} \left( \frac{\nabla u_\Omega}{\sqrt{1 + |\nabla u_\Omega|^2}} \right) = \lambda g(x, u_\Omega). \text{ in } \Omega \quad (1.18)$$

Substitute (1.15) into (1.16) and (1.17). Then we have

$$\frac{\nabla u_\Omega \cdot D^* u_\Omega}{\sqrt{1 + |\nabla u_\Omega|^2}} = |D^* u_\Omega| \text{ in } \Omega, \quad (1.19)$$

$$\frac{(\nu \cdot \nabla u_\Omega) u_\Gamma}{\sqrt{1 + |\nabla u_\Omega|^2}} = -|u_\Gamma| \text{ on } \Gamma. \quad (1.20)$$

These are the equations on the singular parts in $\Omega$ and on $\Gamma$.

Therefore, in the case of $D^* u_\Omega = 0$, the critical point $u \in X$ has the interior regularity that $u = u_\Omega$ belongs to $W^{1,1}(\Omega)$ and it satisfies (1.18). And the boundary condition (1.20) implies

$$u_\Gamma = 0 \text{ or } (\nu \cdot \nabla u_\Omega) = \begin{cases} -\infty & \text{if } u_\Omega > 0, \\ +\infty & \text{if } u_\Omega < 0, \end{cases}$$

because $\frac{(\nu \cdot \nabla u_\Omega)}{\sqrt{1 + |\nabla u_\Omega|^2}} = -\text{sgn} u_\Gamma$ means $|\nu \cdot \nabla u_\Omega| = |\nabla u_\Omega| = +\infty$.

F. Demengel and R. Temam [3] have obtained the similar expression as in Theorem 1.2 for the subdifferential related to the minimal surface operator. For the similar expression on the subdifferential related to 1-Laplace operator, see B. Kawohl and F. Schuricht [9] and F. Demengel [4].

## 2 The area functional and its convex conjugate

Let $a^*$ be the convex conjugate of $a$ in $\mathbb{R}^N$, namely,

$$a^*(p) = \sup\{p \cdot v - a(v) : v \in \mathbb{R}^N\} \text{ for } p \in \mathbb{R}^N.$$  

Observe that

$$a^*(p) = \begin{cases} 1 - \sqrt{1 - |p|^2} & \text{if } |p| \leq 1, \\ +\infty & \text{otherwise.} \end{cases} \quad (2.21)$$

Let $BV_c(\mathcal{B})$ be all of functions in $BV(\mathcal{B})$ with compact support in $\mathcal{B}$. 
Lemma 2.1. For \( u \in BV_c(\mathcal{B}) \),
\[
\mathcal{A}(u) = \sup \{ \int_{\mathcal{B}} u \mathrm{div} p \, dx - \int_{\mathcal{B}} a^*(p) \, dx : p \in V \},
\]
(2.22)
where
\[
V = \{ p \in L^1(\mathcal{B} : \mathbb{R}^N) : \mathrm{div} p \in L^N(\mathcal{B}) \}.
\]
(2.23)

Proof.
1.
\[
\mathcal{A}(u) = \sup \{ \int_{\mathcal{B}} u \mathrm{div} \eta \, dx - \int_{\mathcal{B}} a^*(\eta) \, dx : \eta \in C^1_c(\mathcal{B}; \mathbb{R}^N), |\eta| \leq 1 \}.
\]
(2.24)
Indeed, the condition \( |\hat{\eta}| \leq 1 \) in (1.5) implies
\[
|\eta_0| \leq \sqrt{1 - |\eta|^2}, \quad |\eta| \leq 1.
\]
Thus \( \int_{\mathcal{B}} \eta_0 \, dx \leq \int_{\mathcal{B}} \sqrt{1 - |\eta|^2} \, dx \) and the right-hand side of (1.5)(with \( U = \mathcal{B} \)) is dominated by that of (2.24). On the other hand, the continuous function
\[
0 \leq \sqrt{1 - |\eta|^2} \leq 1 \text{ on } \mathcal{B}
\]
can be approximated by \( C^0_0 \)-function \( \xi \) with \( 0 \leq \xi \leq \sqrt{1 - |\eta|^2} \) in \( L^1 \) sense. Thus (2.24) holds.
2. Since \( C^1_c(\mathcal{B}; \mathbb{R}^N) \subset V \),
\[
\mathcal{A}(u) \leq \sup \{ \int_{\mathcal{B}} u \mathrm{div} p \, dx - \int_{\mathcal{B}} a^*(p) \, dx : p \in V \}.
\]
(2.25)
3. By (2.21), \( \int_{\mathcal{B}} a^*(p) \, dx = +\infty \) if \( a^* \circ p \notin L^1(\mathcal{B}) \). Therefor, \( V \) in the right-hand side of (2.25) can be replaced by \( V_1 \):
\[
V_1 = \{ p \in L^1(\mathcal{B} ; \mathbb{R}^N) : a^* \circ p \in L^1(\mathcal{B}), \mathrm{div} p \in L^N(\mathcal{B}) \}
\]
\[
= \{ p \in L^\infty(\mathcal{B} ; \mathbb{R}^N) : |p|_\infty \leq 1, \mathrm{div} p \in L^N(\mathcal{B}) \}
\]
(2.26)
Since \( u \) has a compact support \( K \equiv \text{supp}[u] \),
\[
\int_{\mathcal{B}} u \mathrm{div} p \, dx = \int_K u \mathrm{div} p \, dx.
\]
Let \( O_1, O_2 \) be open sets with \( K \subset O_1 \subset O_2 \subset \mathcal{B} \) and \( \psi \) be \( C^\infty \)-function with \( 0 \leq \psi \leq 1, \psi = 1 \) on \( O_1 \) and \( \psi = 0 \) outside \( O_2 \). Then \( \mathrm{div}(\psi p) = \mathrm{div} p \) on \( K \) and \( |\psi p| \leq |p| \). By the monotonicity of \( a^*(p) \) with respect to \( |p| \),
\[
\mathcal{A}(u) = \sup \{ \int_{\mathcal{B}} u \mathrm{div} p \, dx - \int_{\mathcal{B}} a^*(p) \, dx : p \in V \}
\]
\[
= \sup \{ \int_{\mathcal{B}} u \mathrm{div} p \, dx - \int_{\mathcal{B}} a^*(p) \, dx : p \in V_1, \text{ suppp } \subset \mathcal{B} \}.
\]
4. To complete proof, it is sufficient to approximate the above integrals by ones whose $p$ is replaced by functions $\eta \in C^1_c(\mathbb{R}; \mathbb{R}^N)$ with $|\eta| \leq 1$. Let $\rho_\epsilon$ be a standard mollifier and set $p_\epsilon = \rho_\epsilon \ast p$. Then $p_\epsilon \in C^1_c(\mathbb{R}; \mathbb{R}^N)$ with $|p_\epsilon| \leq 1$ for small $\epsilon > 0$ and

$$
p_{\epsilon_j} \to p \quad \text{a.e.,}
$$

$$
div p_{\epsilon_j} = \rho_{\epsilon_j} \ast \text{div} p \to \text{div} p \quad \text{in } L^N(\mathcal{B}) ,
$$
as $\epsilon_j \to 0$. Thus

$$
\int_{\mathcal{B}} u \text{div} p_{\epsilon_j} \, dx \to \int_{\mathcal{B}} u \text{div} p \, dx,
$$
and, by the Lebesgue’s dominated convergence theorem,

$$
\int_{\mathcal{B}} a^*(p_{\epsilon_j}) \, dx \to \int_{\mathcal{B}} a^*(p) \, dx.
\square
$$

**Remark.** Set

$$W = \{p \in L^\infty(\mathcal{B}; \mathbb{R}^N) : \text{div} p \in L^N(\mathcal{B})\} .$$

Since $V_1 \subset W \subset V$, as we state in the proof, for $u \in BV_c(\mathcal{B})$ we have

$$
\mathcal{A}(u) = \sup \left\{ \int_{\mathcal{B}} u \text{div} p \, dx - \int_{\mathcal{B}} a^*(p) \, dx : p \in V \right\}
$$

$$
= \sup \left\{ \int_{\mathcal{B}} u \text{div} p \, dx - \int_{\mathcal{B}} a^*(p) \, dx : p \in W \right\}
$$

$$
= \sup \left\{ \int_{\mathcal{B}} u \text{div} p \, dx - \int_{\mathcal{B}} a^*(p) \, dx : p \in V_1 \right\}
$$

and we can impose $p$ on its compact support in $\mathcal{B}$. Moreover, since $a^*$ is even, the above integral $\int_{\mathcal{B}} u \text{div} p \, dx$ can be replaced by $\pm \int_{\mathcal{B}} u \text{div} p \, dx$.

Let $U$ be a bounded domain in $\mathbb{R}^N$. For any $u \in BV(U)$ and any $p \in W(U) = \{p \in L^\infty(U; \mathbb{R}^N) : \text{div} p \in L^N(U)\}$, define the distribution $p \cdot Du$ by

$$
\int_U \varphi (p \cdot Du) = - \int_U (\text{div} p) u \varphi \, dx - \int_U (p \cdot \nabla \varphi) u \, dx \quad (2.27)
$$

for all $\varphi \in C_0^\infty(U)$. R. Kohn and R. Temam ([10] Proposition 1.1) have shown that $p \cdot Du$ is a bounded measure with $|p \cdot Du| \leq ||p||_{\infty} |Du|$ and obtained the Green’s formula:

$$
\int_{\partial U} (p \cdot \nu) u \nu \phi d\mathcal{H}^{N-1} = \int_U (p \cdot Du) \phi + \int_U (\text{div} p) u \phi \, dx + \int_U (p \cdot \nabla \phi) u \, dx \quad (2.28)
$$

for all $\phi \in C^1_c(U)$.
For \( u \in BV_c(\mathcal{B}) \) and \( p \in W \), define \( p \cdot Du \) by (2.27) with \( U = \mathcal{B} \). Choosing \( \varphi \in C^\infty(\mathcal{B}) \) with \( \varphi = 1 \) on \( \text{supp} u \), we have
\[
\int_{\mathcal{B}} p \cdot Du = - \int_{\mathcal{B}} (\text{div} p) u dx. \tag{2.29}
\]

For \( u \in X \), the restriction \( u_\Omega \) of \( u \) on \( \Omega \) belongs to \( BV(\Omega) \). Defining \( p \cdot Du_\Omega \) by (2.27) with \( U = \Omega \) and \( u = u_\Omega \), we have
\[
\int_{\partial \Gamma} (p \cdot \nu) u_\Gamma \varphi d\mathcal{H}^{N-1} - \int_{\Omega} (p \cdot Du_\Omega) \varphi dx
= \int_{\Omega} (p \cdot \text{div} u_\Omega) \varphi dx + \int_{\Omega} (p \cdot \nabla \varphi) u_\Omega dx. \tag{2.30}
\]

Any \( \varphi \in C^\infty(\mathcal{B}) \) can be regard as \( \varphi \in C^1(\Omega) \) and
\[
\int_{\mathcal{B}} (\text{div} u) \varphi dx + \int_{\mathcal{B}} (p \cdot \nabla \varphi) u dx = \int_{\Omega} (\text{div} u_\Omega) \varphi dx + \int_{\Omega} (p \cdot \nabla \varphi) u_\Omega dx.
\]

By (2.27)(with \( U = \mathcal{B} \)), (2.30) and the above equation, we have
\[
\int_{\mathcal{B}} \varphi (p \cdot Du) = \int_{\Gamma} (p \cdot \nu) u_\Gamma \varphi d\mathcal{H}^{N-1} - \int_{\Omega} (p \cdot Du_\Omega) \varphi dx
\]
for all \( \varphi \in C^\infty(\mathcal{B}) \). Thus,
\[
p \cdot Du = p \cdot Du_\Omega - (p \cdot \nu) u_\Gamma d\mathcal{H}^{N-1} \tag{2.31}
\]
for \( u \in X \).

**Remark.** Choose \( p = e_i \ (1 \leq i \leq N) \) in the canonical basis \( \{e_1, \cdots, e_N\} \) of \( \mathbb{R}^N \). Then (2.31) yields (1.13).

**Remark 2.2.** By (2.29) and the remark given after the proof of Lemma 2.1,
\[
\mathcal{A}(u) = \sup \{ \pm \int_{\mathcal{B}} u \text{div} p dx - \int_{\mathcal{B}} a^*(p) dx : p \in Y \}
= \sup \{ \pm \int_{\mathcal{B}} p \cdot Du - \int_{\mathcal{B}} a^*(p) dx : p \in Y \} \tag{2.32}
\]
for \( u \in BV_c(\mathcal{B}) \), where \( Y = V_1, W, V_{1,c} = \{ p \in V_1 : \text{supp}[p] \subset \mathcal{B} \} \) or \( W_c = \{ p \in W : \text{supp}[p] \subset \mathcal{B} \} \).

Define a functional \( \mathcal{A}^* \) on \( W \) by
\[
\mathcal{A}^*(p) = \int_{\mathcal{B}} a^*(p) dx \tag{2.33}
\]
for \( p \in W \). Then
\[
\mathcal{A}^*(p) \begin{cases} 
\leq |\mathcal{B}| & \text{if } p \in V_1, \\
= \infty & \text{otherwise.}
\end{cases}
\]

Since the right-hand side of (2.32) is determined ony by \( Du \), we denote it by \( \mathcal{B}(Du) \).
Definition 2.3. For \( u \in BV_c(\mathcal{B}) \), Define a functional \( B \) of the Radon measure \( Du \) by
\[
B(Du) = \sup \left\{ \int_B p \cdot Du - A^*(p) : p \in V_1 \right\}. \tag{2.34}
\]

Remark 2.4. By (2.32), of course,
\[
B(Du) = A(u) = \sup \left\{ - \int_B u div p \, dx - A^*(p) : p \in Y \right\} = \sup \left\{ \int_B p \cdot Du - A^*(p) : p \in Y \right\}
\]
where \( Y = V_1, V_{1,c}, W \) or \( W_c \).

For a Radon measure \( \mu = \mu^a + \mu^s \), where \( \mu^a = f(x)dx \), \( \mu^s \) be respectively the absolute continuous and singular parts with respect to the Lebesgue measure, define a measure \( a(\mu) \) by \( a(\mu) = a(f(x))dx + a_\infty(\mu^s) \), where \( a_\infty(x) = \lim_{t \to \infty} a(tx)/t = |x| \), i.e.,
\[
a(\mu) = a(f(x))dx + |\mu^s|.
\]

F. Demengel and R. Temam [3] have shown
\[
\int_{\mathcal{B}} \varphi a(\mu) = \sup \left\{ \int_{\mathcal{B}} \varphi p \, d\mu - \int_{\mathcal{B}} \varphi a^*(p) dx : p \in V_1 \right\}
\]
for all \( \varphi \in C^\infty(\mathcal{B}) \) with \( \varphi \geq 0 \). Let \( u \in BV_c(\mathcal{B}) \). Put \( \mu = Du = \nabla u \, dx + D^s u \).

Choosing \( \varphi \) with \( \varphi = 1 \) on supp\([u]\), we have
\[
\int_{\mathcal{B}} a(Du) = \sup \left\{ \int_{\mathcal{B}} p \cdot Du - \int_{\mathcal{B}} a^*(p) dx : p \in V_1 \right\}.
\]
Thus
\[
\int_{\mathcal{B}} a(Du) = \int_{\mathcal{B}} a(\nabla u) dx + |D^s u|(\mathcal{B}) = B(Du) = A(u) \tag{2.35}
\]
for \( u \in BV_c(\mathcal{B}) \). This shows that the equation (1.7) holds for the measure \( a(Du) \) defined above.

Denote by \([Du]_\Omega\) the restriction of \( Du \) on \( \Omega \). By the monotonicity of \( a(v) \) with respect to \(|v|\) and the above equation,
\[
B([Du]_\Omega) \leq B(Du).
\]
For $h \in W^{1,1}(\mathcal{Q})$ with compact support, we write simply $B(\nabla h) = B(\nabla h dx)$, i.e.,

$$B(\nabla h) = \int_{\mathcal{Q}} a(\nabla h) dx.$$  

**Lemma 2.5.** Let $p \in W$, then

$$\int_{\mathcal{Q}} a^*(p) dx = \sup\{-\int_{\mathcal{Q}} u \nabla p dx - B(Du) : u \in X\}  \quad (2.36)$$

**Remark.** (2.36) hold even if the left-hand side equals to $+\infty$.

**Proof of Lemma 2.5.**

1. Fix any $u \in X$. Let $\varphi \in C_0^\infty(\mathcal{Q})$ satisfying $0 \leq \varphi \leq 1$ and $\varphi = 1$ on $\overline{\Omega}$, then $\varphi p \in W$ for all $p \in W$. Since $\text{supp}[Du] \subset \overline{\Omega}$, by the definition 2.3 and Remark 2.4, we have

$$B(Du) \geq \int_{\mathcal{Q}} p \cdot Du - \int_{\mathcal{Q}} a^*(\varphi p) dx,$$

for all $p \in W$. Letting $\varphi \searrow \chi_{\mathcal{Q}}$ yields

$$B(Du) \geq \int_{\mathcal{Q}} p \cdot Du - \int_{\mathcal{Q}} a^*(p) dx,$$

Thus,

$$\int_{\mathcal{Q}} a^*(p) dx \geq \sup\{\int_{\mathcal{Q}} p \cdot Du - B(Du) : u \in X\}$$

$$= \sup\{-\int_{\mathcal{Q}} u \nabla p dx - B(Du) : u \in X\}  \quad (2.37)$$

for all $p \in W$.

2. Define a functional $A$ on $L^1(\mathcal{Q}; \mathbb{R}^N)$ by

$$A(g) = \int_{\mathcal{Q}} a(g) dx \quad \text{for} \ g \in L^1(\mathcal{Q}; \mathbb{R}^N).$$

I. Eckeland and R. Temam have shown that the convex conjugate $A^*$ of $A$ on the dual space $L^\infty(\mathcal{Q}; \mathbb{R}^N)$ is given by $A^*(\cdot) = \int_{\mathcal{Q}} a^*(\cdot) dx$ (see Lemma 1.1. Chap. V [6]), namely,

$$\int_{\mathcal{Q}} a^*(q) dx = A^*(q)$$

$$= \sup\{\int_{\mathcal{Q}} q \cdot g dx - \int_{\mathcal{Q}} a(g) dx : g \in L^1(\mathcal{Q}; \mathbb{R}^N)\}$$
for all \( q \in L^\infty(\mathcal{B}; \mathbb{R}^N) \). Set \( q = \chi_\Omega p \), then

\[
\int_\Omega a^*(p)dx = \sup\{ \int_\Omega p \cdot gdx - \int_\Omega a(g)dx : g \in L^1(\mathcal{B}; \mathbb{R}^N) \}
\]

Note that, for \( g \in L^1(\mathcal{B}; \mathbb{R}^N) \), \( \chi_\Omega g \in L^1(\mathcal{B}; \mathbb{R}^N) \) and \( \int_\mathcal{B} a(g)dx \geq \int_\Omega a(\chi_\Omega g)dx = \int_\Omega a(g)dx \). Thus

\[
\int_\Omega a^*(p)dx = \sup\{ \int_\Omega p \cdot gdx - \int_\Omega a(g)dx : g \in L^1(\mathcal{B}; \mathbb{R}^N) \}
\]

(2.37) and (2.38) show the lemma.

Since \( \mathcal{B}(Du) = A(u) \), by Lemma 2.5,

\[
\int_\Omega a^*(p)dx = \sup\{ -\int_\mathcal{B} \text{div} pdx - A(u) : u \in X \}
\]

for \( p \in W \). The right-hand side is depending only on \( \pm \text{div} p \in L^N(\mathcal{B}) \). Set

\[
Z = \{ z \in L^N(\mathcal{B}) : z = -\text{div} p, \ p \in W_c \},
\]

and define a functional \( A^* \) on \( L^N(\mathcal{B}) \) by

\[
A^*(z) = \begin{cases} 
-\int_\mathcal{B} uzd\mu - A(u) : u \in X 
&\text{if } z \in Z \\
+\infty
&\text{otherwise}
\end{cases}
\]

(2.40)

Note that \( A^*(z) = A^*(-p) = A^*(p) = A^*(-z) \).

**Lemma 2.6.** \( A^* \) is lower semicontinuous with respect to \( L^N \)-topology.

**Proof** It is sufficient to show that the set \( \{ z \in L^N(\mathcal{B}) : A^*(z) \leq \alpha \} \) is closed for any \( \alpha \in \mathbb{R} \). Suppose the sequence \( \{ z_n \} \) satisfies \( A^*(z_n) \leq \alpha \) and \( z_n \to z \) in \( L^N(\mathcal{B}) \). Thus, \( z_n = -\text{div} p_n \) and \( |p_n|_\infty \leq 1 \) and we may assume \( \text{supp}[p_n] \subset O \) for an open set \( O \) with \( \Omega \subset O \subset \mathcal{B} \). There is a subsequence \( \{ p_{n'} \} \) and \( p \in L^\infty(\mathcal{B}; \mathbb{R}^N) \) such that

\[
p_{n'} \to p \quad \text{weakly*},
\]
Thus $|p|_{\infty} \leq 1$ and $\text{supp}[p] \subset O$. Since
\[
\int_{\partial B} z_{n'} \varphi dx = -\int_{\partial B} (\text{div} p_{n'}) \varphi dx = \int_{\partial B} p_{n'} \cdot \nabla \varphi dx
\]
and
\[
\int_{\partial B} z_{n'} \varphi dx \to \int_{\partial B} z \varphi dx
\]
for all $\varphi \in C^\infty_c(\partial B)$. Thus $z = -\text{div} p$. Since $z_{n'} \to z$ and
\[
\int_{\partial B} u z_{n'} dx - A(u) \leq \alpha,
\]
\[
\int_{\partial B} u z dx - A(u) = A^*(z) \leq \alpha.
\]
This shows the lemma.

Note that, for any $z \in Z$, if $A^*(z)$ is finite then $z = -\text{div} p$ with $p \in W_c$ and
\[
A^*(z) = \int_{\Omega} a^*(p) dx. \tag{2.41}
\]

3 Proof of Theorem 1.2

Proposition 3.1. Let $u \in X$ and suppose that there exists $z \in L^N(\partial B)$ holding the inequality,
\[
A(v) - A(u) \geq \lambda \int_{\partial B} z(v - u) dx \quad \text{for all } v \in X. \tag{3.42}
\]
Then there exists $p \in V_1$ with $-\text{div} p = z$ such that
\[
p = \frac{\nabla u_{\Omega}}{\sqrt{1 + |\nabla u_{\Omega}|^2}} \quad \text{in } \Omega, \tag{3.43}
\]
\[
p \cdot D^s u_{\Omega} = |D^s u_{\Omega}| \quad \text{in } \Omega, \tag{3.44}
\]
\[
(\nu \cdot p) u_{\Gamma} = -|u_{\Gamma}| \quad \text{on } \Gamma. \tag{3.45}
\]

Proof.
1. We may assume that $z$ in (3.42) has a compact support in $\partial B$. Indeed, if not, by applying Lemma 4.1 in Appendix, we have $p \in W_c$ satisfying $z = -\text{div} p$
on $\Omega$. Thus we can replace $z$ in the inequality by $\hat{z} \equiv -\text{div} p$ which has a compact support.

2. Thus there exists $z = -\text{div} p$ with $p$ in $W_c$ such that

$$\int_B z dx - A(u) \geq \int_B z dx - A(v) \quad \text{for all } v \in X.$$ 

Since $u \in X$, this implies

$$\int_B z dx - A(u) = \sup\{ \int_B z dx - A(v) : v \in X \}$$

$$= \sup\{ -\int_B \text{div} p dx - A(v) : v \in X \}$$

$$= A^*(p) = A^*(\hat{z}).$$

The left-hand side is finite and so is $A^*(p)$. Hence, $p \in V_{1,c}$ and by (2.41)

$$-\int_B u \text{div} p dx - A(u) = \int_{\Omega} a^*(p) dx,$$

$$\int_B p \cdot D u dx - \int_B a(Du) = \int_{\Omega} a^*(p) dx.$$

Note that the domain $\mathcal{B}$ of integration in the left-hand side can be regarded as $\Omega$ because the support of $Du$ is in it. Since $Du = Du_\Omega - u_\Gamma \nu dH^{n-1}$ and $Du_\Omega = \nabla u_\Omega dx + D^* u_\Omega$, we have

$$\int_{\Omega} p \cdot \nabla u_\Omega dx + \int_{\Omega} p \cdot D^* u_\Omega dx - \int_{\Gamma} (p \cdot \nu) u_\Gamma dH^{N-1}$$

$$- \int_{\Omega} a(\nabla u_\Omega) dx - \int_{\Omega} |D^* u_\Omega| dx - \int_{\Gamma} |u_\Gamma| dH^{N-1}$$

$$= \int_{\Omega} a^*(p) dx,$$

$$0 = \int_{\Omega} (a^*(p) - ((p \cdot \nabla u_\Omega) - a(\nabla u_\Omega))) dx$$

$$+ \int_{\Omega} (|D^* u_\Omega| - (p \cdot D^* u_\Omega))$$

$$+ \int_{\Gamma} (|u_\Gamma| + (p \cdot \nu) u_\Gamma) dH^{N-1}.$$ 

Note that each integrand of the three terms of the last equation is nonnegative. Hence the equation implies the integrands vanish identically.

$$a^*(p) = p \cdot \nabla u_\Omega - a(\nabla u_\Omega) \quad (3.46)$$

$$p \cdot D^* u_\Omega = |D^* u_\Omega|$$

$$(p \cdot \nu) u_\Gamma = - |u_\Gamma|$$
implies \( p = a'(\nabla u_\Omega) \) which is equal to (3.43). Other two equations are (3.44) and (3.45).

\[\Box\]

**Proof of Theorem 1.2**

Since \( u \in X \) is a solution of variational inequality (1.11), setting \( z = g(x, u) \) and noting \( \text{supp}[g(x, u)] \subseteq \Omega \), apply Proposition 3.1.

\[\Box\]

**Remark 3.2.** Finally, we illustrate an analogy between our approach developed here and the process in the theoretical mechanics deriving the Hamilton’s canonical form from the Euler-Lagrange equation. In what follows, we often describe formally without the mathematical accuracy.

Let regard our functional \( L_\lambda \) as a Lagrangean of the form

\[ L_\lambda(u, Du) = B(Du) - \lambda G(u), \tag{3.47} \]

In the case of a smooth \( u \), its Euler-Lagrange equation is given by the second order differential equation in (0.1). For the functional \( L_\lambda \) involving the Radon measure \( Du \), we cannot derive directly such an equation. By using the Legendre transformation

\[ A^*(p) = \sup \left\{ \int_\Omega p \cdot Du \, dx - B(Du) : u \in X \right\}, \tag{3.48} \]

the equation in which \( A^*(p) \) replaced \( \int_\Omega u^*(p)dx \) is given in Lemma 2.5, and setting the Hamiltonian

\[ H(u, p) = A^*(p) - \lambda G(u), \tag{3.49} \]

we introduce a system of the first order differential equations

\[
\begin{align*}
Du &= -\frac{\partial H}{\partial p} = -\frac{\partial A^*}{\partial p}, \\
D \cdot p &= \frac{\partial H}{\partial u} = -\lambda \frac{\partial G}{\partial u}.
\end{align*}
\tag{3.50}
\]

In the theoretical dynamics, the canonical form governing the motion of a mass has the same type as (3.50) where \( u \) and \( p = B'(Du) \) represent the position and the momentum of the mass respectively and \( D = \frac{d}{dt} \). As we stated in the above, (3.48) is written formally

\[ A^*(z) = \sup \{ \langle z, u \rangle - A(u) : u \in X \} \]

where \( A(u) = B(Du), \) \( z = -\text{div}p \). The variational inequality (1.11) implies

\[ \lambda \int_\Omega gudx - A(u) = A^*(\lambda g) = A^*(p) \]
with $\lambda g = -\div p$. By Lemma 2.5 (the key lemma),

$$\int_{\Omega} a^*(p) dx = \int_{\Omega} (-\div p) u dx - A(u)$$

By using $Du = \nabla u_\Omega dx + D^* u_\Omega - u_\Gamma u dH^{n-1}$,

$$\int_{\Omega} (a^*(p) - p \cdot \nabla u_\Omega - a(\nabla u_\Omega)) dx$$
$$+ \int_{\Omega} (|Du| - p \cdot D^* u)$$
$$+ \int_{\Gamma} (|u_\Gamma| + (p \cdot \nu)) dH^{n-1}$$
$$= 0.$$ 

By this, instead of the formal relation $p = B'(Du)$ between the momentum and the velocity, we obtained more explicit relations (3.43)-(3.45) in Proposition 3.1.

Since $\mathcal{A}^*(p) = \int_{\Omega} a^*(p) dx$ and $G(u) = \int_{\Omega} G(x,u) dx$,

$$< \frac{\partial \mathcal{A}^*}{\partial p}, \phi > = \int_{\Omega} (a^*)'(p) \cdot \phi dx = - \int_{\Omega} \frac{p \cdot \phi}{\sqrt{1-|p|^2}} dx,$$
$$< \frac{\partial G}{\partial u}, \varphi > = \int_{\Omega} g(x,u) \varphi dx,$$

we can write the equations in (3.50) in weak form:

$$< u, -\div \phi >= < \frac{p}{\sqrt{1-|p|^2}}, \phi > \text{ for all } \phi \in C_c^{\infty}(\mathcal{B};\mathbb{R}^N),$$
$$< p, \nabla \varphi > = - \lambda < g, \varphi > \text{ for all } \varphi \in C_c^{\infty}(\mathcal{B};\mathbb{R}).$$

Hence

$$Du = \nabla u_\Omega = \frac{p}{\sqrt{1-|p|^2}},$$

which is equivalent (3.43), and

$$\div p = -\lambda g(x,u).$$

4 Appendix.

**Lemma 4.1.** For any $z \in L^N(\mathcal{B})$, there exists $\hat{z} \in L^N(\mathcal{B})$ with compact support such that $z = \hat{z}$ on $\Omega$ and $\hat{z} = -\div p$ for some $p \in L^\infty(\mathcal{B};\mathbb{R}^N)$ with compact support in $\mathcal{B}$. 

Proof.

Let $O_1$, $O_2$ be open sets satisfying $\Omega \subset O_1 \subset O_2 \subset B$, and $\psi$ be a $C^\infty_c$-function such that $0 \leq \psi \leq 1$, $\psi = 1$ on $\overline{O_1}$ and $\psi = 0$ outside $O_2$. Let $l_i$ be a line parallel to $x_i$-axis, say, $l_i = \{(x_1, \cdots, x_{i-1}, \xi, x_{i+1}, \cdots, x_N) : \xi \in \mathbb{R}\}$. For each $y_i = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_N) \in \mathbb{R}^{N-1}$, we write simply $l_i = \{((\xi, y_i) : \xi \in \mathbb{R}\}$. The line segment $l_i \cap B$ can be expressed by $l_i$ with $|\xi| < R_i$ for some $R_i = R_i(y_i) > 0$. Set $z_0 = \frac{1}{N} \psi z$ and

$$w_i(x) = -\int_{-R_i}^{x_i} z_0(\xi, y_i) d\xi$$

for $|x_i| \leq R_i$ and $1 \leq i \leq N$. Then

$$w_i \in L^\infty(B), \quad \frac{\partial w_i}{\partial x_i} = -z_0.$$

Denote $w = (w_1, \cdots, w_N)$ and set $p = \psi w$, then $p$ has a support in $\overline{O_2}$, belongs to $L^\infty(B; \mathbb{R}^N)$ and is equal to $w$ in $\overline{O_1}$. Hence, $\hat{z} = -\text{div} p$ has a compact support in $\overline{O_2}$ and

$$\hat{z} = -\text{div} p = N z_0 = z \quad \text{in} \ O_1.$$

□

References


