An algorithm for computing Grothendieck local residues I
— shape basis case —

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In this talk, we will give an algorithm for exactly computing Grothendieck local residues for rational \( n \)-forms of \( n \) variables under certain condition and show an implementation on a computer algebra system Risa/Asir. Grothendieck local residue is natural generalization of the well-known residue for complex functions of one variable and is defined as an integration of meromorphic \( n \)-form of complex \( n \) variables on a real \( n \)-cycle around an isolated common zero. Let us recall the analytic definition of Grothendieck local residues. (see [1] chapter 5 for detail.)

**Definition.** Denote by \( \mathcal{O}(U) \) a ring of holomorphic functions on a ball \( U \subset \mathbb{C}^n \). Suppose that \( f_1(x), \ldots, f_n(x) \in \mathcal{O}(U) \) make regular sequence and have only one isolated common zero \( b \in U \). Let \( \Gamma(b) \) be a real \( n \)-cycle around \( b \) defined by \( \Gamma(b) = \{ x \in U | \|f_1(x)\| = \varepsilon, \ldots, \|f_n(x)\| = \varepsilon \} \) and oriented by \( d(\arg f_1) \wedge \cdots \wedge d(\arg f_n) \geq 0 \). Denote \( \tau_F = (f_1(x) \cdots f_n(x))^{-1} dx_1 \wedge \cdots \wedge dx_n \), where \( x = (x_1, \ldots, x_n) \).

For any \( \varphi(x) \in \mathcal{O}(U) \), the integration

\[
\text{Res}_b(\varphi(x) \tau_F) = \left( \frac{1}{2\pi} \right)^n \int_{\Gamma(b)} \varphi(x) \tau_F
\]

is called the Grothendieck local residue of meromorphic \( n \)-form \( \varphi(x) \tau_F \).

Grothendieck local residue is a quite important concept in pure mathematics. However it is hard to directly evaluate them from the definition because of complicated geometric shape of the real \( n \)-cycle in \( 2n \)-dimensional real space. The correspondence \( \varphi \mapsto \text{Res}_b(\varphi \tau_F) \) given by the local residue can be regarded as a distribution on \( \mathcal{O}(U) \) and can be expressed by a linear partial differential operator. That is, there exists a linear partial differential operator \( T_F = \sum c_\alpha(x) \partial^\alpha \) determined by the regular sequence \( F = \{ f_1, \ldots, f_n \} \) such that \( \text{Res}_b(\varphi \tau_F) = (T_F \bullet \varphi)_{x=b} \). Here “\( \bullet \)” is notation to express action by a differential operator to a function. Thus, the local residue can be evaluated if the operator \( T_F \) can be calculated. Our purpose is to develop new and effective method for exactly computing the operator \( T_F \) from the regular sequence under certain condition.

To treat the local residue using computer algebra system, we suppose that the regular sequence consists of polynomials. The set \( F \) generates a zero-dimensional
ideal $I$ in $\mathbb{C}[x] = \mathbb{C}[x_1, \ldots, x_n]$. Then the local residue $\varphi \mapsto \text{Res}_b(\varphi \tau_F)$ is determined by the algebraic local cohomology class $\sigma_F = \left[ \frac{1}{f_1 \cdots f_n} \right] \in H^p_{\mathfrak{m}}(\mathbb{C}[x])$. The linear partial differential operator $T_F$ is called Noether differential operator with respect to the algebraic local cohomology class $\sigma_F$.

As it is well known, a polynomial ideal is decomposed to an intersection of primary ideals. Then the algebraic local cohomology class is also expressed as

$$\sigma_F = \sigma_{F,1} + \cdots + \sigma_{F,\lambda} + \cdots + \sigma_{F,N},$$

where the support $Z_\lambda$ of $\sigma_{F,\lambda}$ coincides the zero set of corresponding primary component of $I$. Let $\beta \in Z_\lambda$ and $\varphi(x) \in \mathbb{C}[x]$. Since $\sigma_F dx = \sigma_{F,\lambda} dx$ on $Z_\lambda$, we have $\text{Res}_b(\varphi \sigma_F dx) = \text{Res}_b(\varphi \sigma_{F,\lambda} dx)$. Thus is allows to compute expression of the local residue on each irreducible components. We denote by $T_{F,\lambda}$ the corresponding Noether differential operator to the local residue $\varphi \mapsto \text{Res}_b(\varphi \sigma_{F,\lambda} dx)$. Hence the set $\{(T_{F,\lambda}, Z_\lambda) \mid \lambda = 1, 2, \ldots, N\}$ gives an expression of the Noether differential operator $T_F$.

In this talk, we treat the special case that the primary ideal $I_\lambda$ is expressed by shape bases. Our purpose is to determine the differential operator $T_{F,\lambda}$ from $I_\lambda$. We use two tools to solve this problem. One is Noether differential operator bases which describes a relation between $I_\lambda$ and $\sqrt{I_\lambda}$. Another is a suitable subset of the annihilating ideal $\text{Ann}_{D_n}(\sigma_{F,\lambda})$ of the algebraic local cohomology class $\sigma_F$. The annihilating ideal is a left ideal in the Weyl algebra $D_n$. So the cost of computation of $\text{Ann}_{D_n}(\sigma_{F,\lambda})$ is high in general.

Under the shape base condition of primary ideals, we can explicitly construct Noether differential operator bases and suitable subset of $\text{Ann}_{D_n}(\sigma_{F,\lambda})$ without Gröbner bases in Weyl algebra. Hence our algorithm is effective.

References

