Computation methods of \(b\)-functions

associated with \(\mu\)-constant deformations

– Case of inner modality 2 –

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In this talk, computation methods of parametric \(b\)-functions are introduced for \(\mu\)-constant deformation of quasihomogeneous singularities. The methods of \(b\)-functions associated with \(\mu\)-constant deformations are constructed by using comprehensive Gröbner systems and the set of candidates of roots. In the cases of inner modality 2 ([7]), all \(b\)-functions of associated with \(\mu\)-constant deformations, can be obtained by our computation methods.

Let \(\mathbb{C}(x, \partial_x)\) denote the Weyl algebra, the ring of linear partial differential operators with coefficients in \(\mathbb{C}\), where \(x = (x_1, \ldots, x_n)\), \(\partial_x = (\partial_1, \ldots, \partial_n)\), \(\partial_i = \frac{\partial}{\partial x_i}\).

Let \(f\) be a non-constant polynomial in \(\mathbb{C}[x]\). Then, the annihilating ideal of \(f^s\) is \(\text{Ann}(f^s) := \{ p \in \mathbb{C}(s, x, \partial_x)| pf^s = 0 \}\) where \(s\) is an indeterminate. The \(b\)-function or the Bernstein-Sato polynomial of \(f\) is defined as the monic generator \(b_f(s)\) of \((\text{Ann}(f^s) + \text{Id}(f)) \cap \mathbb{C}[s]\) where \(\text{Id}(f)\) is the ideal generated by \(f\). It is known that the \(b\)-function of \(f\) always has \(s + 1\) as a factor and has a form \((s + 1)b_f(s)\), where \(b_f(s) \in \mathbb{C}[s]\). The polynomial \(b_f(s)\) is called the reduced \(b\)-function of \(f\).

It is known that a basis of the ideal \(\text{Ann}(f^s)\) can be computed by utilizing a Gröbner basis in \(\mathbb{C}(x, \partial_x)\) or PWB algebra ([5]). Moreover, the reduced \(b\)-function \(\tilde{b}_f(s)\) can be obtained by computing a Gröbner basis of \(\text{Ann}(f^s) + \text{Id}(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})\).

Let \(f\) be a parametric polynomial in \((\mathbb{C}[u])[x]\) where \(u = (u_1, \ldots, u_m)\) and \(u\) are parameters. In our previous paper [4], a computation method of comprehensive Gröbner systems (CGS) has been introduced in Poincaré-Birkhoff-Witt (PBW) algebras. Thus, theoretically, a CGS of the ideal \(\text{Ann}(f^s)\) can be computed by utilizing the computation method. Moreover, a CGS of the ideal \(\text{Ann}(f^s) + \text{Id}(f)\) can be computed, too. Hence, parametric \(b\)-functions can be computed by the following algorithm.

**Algorithm 1.**

**Input:** \(f\): a parametric polynomial.

**Output:** reduced \(b\)-functions of \(f\).

**STEP 1:** Compute a CGS of \(\text{Ann}(f^s)\).

**STEP 2:** Compute a CGS of \(\text{Ann}(f^s) + \text{Id}(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})\).

Algorithm 1 has been implemented in the computer algebra system Risa/Asir.

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**Table 1:** reduced \(b\)-functions of \(x^2z + yz^2 + y^6 + u_1y^4z + u_2z^3\)

<table>
<thead>
<tr>
<th>strata</th>
<th>reduced (b)-function</th>
</tr>
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<tbody>
<tr>
<td>(\mathbb{C}^2\setminus\mathbb{V}(u_1))</td>
<td>(B(s)(s + \frac{1}{u_1}))(\mathbb{V}(u_1))</td>
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<tr>
<td>(\mathbb{V}(u_1)\setminus\mathbb{V}(u_1, u_2))</td>
<td>(B(s)(s + \frac{1}{u_1})(s + \frac{1}{u_2}))</td>
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The Milnor number $\mu$ of the singularity $x^2z + yz^2 + y^6 = 0$ is 17 ($S_{17}$ singularity, the inner modality is 2), and the $\mu$-constant deformation is given by $f = x^2z + yz^2 + y^6 + u_1y^3z + u_2z^3$ where $u_1, u_2$ are parameters. Our implementation can output Table 1 as the parametric reduced $b$-function of $f$ within 5 hours where

$$B(s) = (s + \frac{5}{2})(s + \frac{3}{2})(s + \frac{11}{8})(s + \frac{13}{8})(s + \frac{15}{8})(s + \frac{17}{8}) \times (s + \frac{25}{21})(s + \frac{29}{21})(s + \frac{41}{21})(s + \frac{61}{21})(s + \frac{63}{21})(s + \frac{67}{21})(s + \frac{69}{21}).$$

Let us consider another example. The Milnor number of $\mu$ of the singularity $x^2z + yz^2 + xy^4 = 0$ is 16 ($S_{16}$ singularity, the inner modality is 2), and the $\mu$-constant deformation is given by $f = x^2z + yz^2 + xy^4 + u_1y^6 + u_2z^3$ where $u_1, u_2$ are parameters. In this case, our implementation of Algorithm 1 cannot return the parametric reduced $b$-function of $f$ within “2 months”. However, the implementation returns a CGS of $\text{Ann}(f^s)$ within 1 day. Thus, we can infer that the computational complexity of $\text{Ann}(f^s)$ + $\text{Id}(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})$ is quite big.

In order to avoid the big computation, Levandovskyy and Martin-Morales [3] have introduced a smart idea. We adopt the idea for computing $b$-functions of $\mu$-constant deformations. However, the idea is not good enough to decide $b$-functions of $\mu$-constant deformations. We need a further computation step that is checking local cohomology solutions of each holonomic $D$-module associated with a root of $\tilde{b}(s) = 0$, to compute $b$-functions of $\mu$-constant deformations.

In this talk, we introduce the further computation step and the new algorithm for computing $b$-functions associated with $\mu$-constant deformations.

Let $f(u, x) = f_0 + g \in (\mathbb{C}[u])[x]$ be a semi-quasihomogeneous polynomial, where $f_0$ is the quasihomogeneous part (or weighted homogeneous part) and $g$ is a linear combination of upper monomials with parameters $u$. Then, $f$ can be regard as a $\mu$-constant deformation of $f_0$ with an isolated singularity at the origin. We have the following classical results.

**Theorem 1** Let $E_{f_0} = \{ \gamma \in \mathbb{Q}|\tilde{b}_{f_0}(\gamma) = 0 \}$ where $\tilde{b}_{f_0}$ is the reduced $b$-function of $f_0$ on the origin. Then, for $e \in \mathbb{C}^m$, the set of roots of $b$-function of $f(e, x)$, on the origin, the set $E_{f(e, x)} = \{ \gamma|\tilde{b}_{f(e, x)}(\gamma) = 0 \}$ becomes a subset of $E = \{ \gamma - \ell \in \mathbb{Q}|\gamma \in E_{f_0}, \ell \in \mathbb{Z}, -n < \gamma - \ell < 0 \}$ where $\mathbb{Z}$ is the set of integers. That is, $E_{f(e, x)} \subset E$, for $e \in \mathbb{C}^m$.

**Theorem 2** Let $f$ be a non-constant polynomial in $\mathbb{C}[x]$, $H$ a basis of $\text{Ann}(f^s)$ in $\mathbb{C}[s, x, \partial_x]$, $\gamma \in \mathbb{Q}$ and $r \in \mathbb{N}$. Let $G$ be a minimal Gröbner basis of $\text{Id}(H \cup \{ f, \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \} \cup \{(s - \gamma)^r\})$ w.r.t. a block term order $\succ$ s.t. $x \cup \partial_x \gg s$. Then, if $(s - \gamma)^r \in G$, $(s - \gamma)^r$ is a factor of the $b$-function of $f$.

The outline of the new algorithm is the following.

**Algorithm 2.**

**Input:** $f$: a parametric polynomial.

**Output:** reduced $b$-functions of $f$.

**STEP 1:** Compute a set $E$ of candidates of roots of $\tilde{b}_f(s) = 0$.

**STEP 2:** Compute a CGS of $\text{Ann}(f^s)$.

**STEP 3:** Compute a minimal Gröbner basis $G$ of $\text{Ann}(f^s) + \text{Id}((s - \gamma)^r, f)$ (or $\text{Id}((s - \gamma)^r, f, \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n})$) in $\mathbb{C}[s][x, \partial_x]$ where $\gamma \in E$ and $r \in \mathbb{N}_{>0}$.

If $(s - \gamma)^r \in G$, then $(s - \gamma)^r$ is a factor of the $b$-function of $f$.

**STEP 4:** For each stratum, check local cohomology solutions of each holonomic $D$-module associated with the root of $\tilde{b}_f(s) = 0$. 
By executing Algorithm 2, we can obtain Table 2 as the parametric reduced $b$-function of $f = x^2z + yz^2 + xy^4 + u_1y^6 + u_2z^3$ within 4 hours where

$$B(s) = \left( s + \frac{16}{17} \right) \left( s + \frac{19}{17} \right) \left( s + \frac{22}{17} \right) \left( s + \frac{24}{17} \right) \left( s + \frac{26}{17} \right) \left( s + \frac{28}{17} \right) \left( s + \frac{30}{17} \right) \left( s + \frac{32}{17} \right).$$

Table 2: reduced $b$-functions of $x^2z + yz^2 + xy^4 + u_1y^6 + u_2z^3$

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In this talk, we present mainly Algorithm 2 and show all $b$-functions of $\mu$-constant deformation of inner modality 2.

**Keywords:** $b$-functions, comprehensive Gröbner systems, local cohomology

**References**


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