RIMS Workshop (Type A)

Applications of Harmonic Maps and Higgs Bundles to Differential Geometry

RIMS Kyoto University and Online (Zoom)

May 28 – June 2, 2022

Supports

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Program

May 28, 2022

14:00-15:15	John Loftin (Rutgers University, USA) Differential geometry of cubic differentials (1)
15:50-16:50	Shoichi Fujimori (Hiroshima University, Japan) Genus three embedded doubly periodic minimal surfaces with parallel ends

May 29, 2022

9:30-10:45	Martin Traizet (University of Tour, France) Gluing constructions via loop group methods
11:10-12:10	Hiroki Fujino (Nagoya University, Japan) Extension of harmonic function beyond discontinuous boundary values
14:00-15:15	Fran Burstall (University of Bath, UK) Harmonic maps, Gauss maps and classical geometry (1)
15:50-16:50	Natsuo Miyatake (Osaka University, Japan) Kazdan-Warner type equations on Riemannian manifolds, the Hermitian-Einstein equation for diagonal metrics on Higgs bundles, and the variational method

May 30, 2022

9:30-10:45	John Loftin (Rutgers University, USA) Differential geometry of cubic differentials (2)
11:10-12:10	Yu Kawakami (Kanazawa University, Japan) Heinz-type mean curvature estimates and its application
12:10-	**Free Discussion**

May 31, 2022

9:30-10:45	Fran Burstall (University of Bath, UK) Harmonic maps, Gauss maps and classical geometry (2)
11:20-12:00	Tadashi Udagawa (Waseda University, Japan) Globality of the DPW construction for Smyth potentials in the case of $SU_{1,1}$
14:00-15:15	Martin Traizet (University of Tour, France) Loop group methods for the non-abelian Hodge correspondence on a 4-punctured sphere
15:50-16:50	Martin Guest (Waseda University, Japan) Toda and Higgs

June 1, 2022

9:30-10:45	Franz Pedit (UMass Amherst, USA & RIMS Kyoto University, Japan) Higgs bundles, affine spheres, Monge-Ampère equations, and SYZ mirror symmetry
11:10-12:10	Isami Koga (Meiji University, Japan) Equivariant harmonic immersions of the complex projective line into the complex Grassmannians of two-planes
14:00-15:15	Charles Ouyang (UMass Amherst, USA) Maximal surfaces in $\mathbb{H}^{2,2}$ and their limits
15:15-15:25	**Closing Address**

June 2, 2022

10:00-16:00	**Free Discussion**

RIMS Workshop (Type A)

Applications of Harmonic Maps and Higgs Bundles to Differential Geometry

RIMS Kyoto University and Online (Zoom)

May 28 – June 2, 2022

Abstracts

Fran Burstall (University of Bath, UK)

Title: Harmonic maps, Gauss maps and classical geometry

Abstract

Harmonic maps find application as the Gauss maps (in an appropriate sense) of solutions of classical variational problems. Constant curvature surfaces in $\mathbb{R}^3$; Willmore surfaces, projectively minimal and Lie sphere minimal surfaces all fit into this picture giving a uniform approach to their integrability and Baecklund transformations. In these talks, I shall survey this topic and indicate a unifying approach to the latter three surface classes via the classical notion of the Laplace invariant of a line congruence.

John Loftin (Rutgers University, USA)

Title: Differential geometry of cubic differentials

Abstract

Cubic differentials are related to differential geometric structures on surfaces with symmetric group a real form of $\mathrm{SL}(3,\mathbb{C})$. In my first lecture, I will discuss some of the geometry and analysis behind four natural cases. We will focus on 2 cases, hyperbolic affine spheres in 3-space and minimal Lagrangian surfaces in the complex hyperbolic plane and explain their relation to Higgs bundles, harmonic maps, and surface group representations.

In the second lecture, I plan to discuss an important case of limits of hyperbolic affine spheres. Pairs (S, U) of a Riemann surface S of genus g and U a cubic differential on S can be used to parametrize the Hitchin component of the space of representations of the fundamental group into $\mathrm{PSL}(3,\mathbb{R})$. For nonzero U, we study the case sU as s approaches infinity. In particular, we show the geometry in this limit can be read off explicitly from U, in terms of an embedding of the universal cover of S into the real building given by the asymptotic cone of the symmetric space $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$. This is based on joint work with Andrea Tamburelli and Mike Wolf.

Martin Traizet (University of Tours, France)

Title (1): Gluing constructions via loop group methods

Abstract

Loop group methods (e.g. the DPW method) allow the construction of constant mean curvature surfaces from holomorphic data. For surfaces with non-trivial topology, a monodromy problem must be solved. I will explain some ideas on how to solve the monodromy problem by an implicit function argument at a point (in the parameter space) where the surface degenerates. With these ideas, we can carry on gluing constructions classically done using PDE methods.

Title (2): Loop group methods for the non-abelian Hodge correspondence on a 4-punctured sphere

Abstract

The non-abelian Hodge correspondence is a real analytic map between the moduli space of Higgs bundles and the De Rham moduli space of flat connections, mediated by solutions of the self-duality equation. Using some of the ideas in the first talk, I will explain how to construct solutions of the self-duality equation for Higgs fields on a 4-punctured sphere with small parabolic weights. This allow us to compute quite explicitly the non-abelian Hodge correspondence in that particular case. Joint work with L. Heller and S. Heller.

Shoichi Fujimori (Hiroshima University, Japan)

Title: Genus three embedded doubly periodic minimal surfaces with parallel ends

Abstract

We construct a one-parameter family of embedded doubly periodic minimal surfaces in Euclidean three space. Each surface of the family has genus three with four parallel ends in the quotient. This is joint work with Peter Connor (Indiana University South Bend) and Phillip Marmorino (University of Notre Dame).

Hiroki Fujino (Nagoya University, Japan)

Title: Extension of harmonic function beyond discontinuous boundary values

Abstract

The classical Schwarz reflection principle induces some refection principles for minimal surfaces in the Euclidean space. Such reflection principles enable us to construct various types of complicated and beautiful periodic minimal surfaces in an intuitive manner.

In this talk, we introduce a method of real analytic extension for harmonic functions beyond their discontinuous boundary values. In addition, as an application of the result, we construct some kinds of periodic zero mean curvature surfaces. This talk is based on the joint research with Shintaro Akamine (Nihon University).

Martin Guest (Waseda University, Japan)

Title: Toda and Higgs

Abstract

The 2D Toda equations can take several forms, most of which can be related to harmonic map equations for maps from a surface to a symmetric space. It turns out that one of these forms can be related to the Higgs equations (Hitchin's equations), and in this situation the DPW potential plays the role of the Higgs field. We have studied in some detail the case where the Higgs field is given by monomials; this is analogous to case of the the Smyth potential in the version of the Toda equations corresponding to CMC surfaces. Takuro Mochizuki has shown that the nonabelian Hodge Correspondence applies in this situation, even when singularities are allowed in the potential (in contrast to the CMC situation, where there are no singularities). We shall discuss some differential geometric properties of the resulting harmonic maps.

Yu Kawakami (Kanazawa University, Japan)

Title: Heinz-type mean curvature estimates and its application

Abstract

We would like to talk about a systematic study of Heinz-type mean curvature estimates in Euclidean space and Lorentz-Minkowski space. In particular, we give a unified description of Heinz-type mean curvature estimates under an assumption on the gradient bound for space-like graphs and time-like graphs in Lorentz-Minkowski space and its application to Bernstein-type theorem of entire graphs of constant mean curvature. This talk is based on a joint work with Atsufumi Honda, Miyuki Koiso and Syunsuke Tori.

Isami Koga (Meiji University, Japan)

Title: Equivariant harmonic immersions of the complex projective line into the complex Grassmannians of two-planes

Abstract

In this talk, I would like to introduce a classification result of equivariant harmonic immersios of complex projective line into the complex Grassmannian of two-planes. To classify them, at first we research some properties of invariant connections on homogeneous vector bundle of rank two over the complex projective line. Then we give the main result.

Natsuo Miyatake (Osaka University, Japan)

Title: Kazdan-Warner type equations on Riemannian manifolds, the Hermitian-Einstein equation for diagonal metrics on Higgs bundles, and the variational method

Abstract

We can consider the Hermitian-Einstein equation for diagonal metrics on Higgs bundles over a Kähler manifold as one of the Kazdan-Warner type PDE in addition to the view that it is the Hermitian-Einstein equation. Here we mean by a diagonal metric a Hermitian metric on a direct sum of holomorphic line bundles diagonal to the decomposition and by Kazdan-Warner equation the Poisson's equation with an exponential non-linear term. In this talk, we formally generalize the Hermitian-Einstein equation for diagonal metrics on Higgs bundles to PDEs on Riemannian manifolds and solve them when the manifold is compact using the variational method inspired by the Kempf-Ness's theorem for linear torus actions.

Charles Ouyang (UMass Amherst, USA)

Title: Maximal surfaces in $\mathbb{H}^{2,2}$ and their limits

Abstract

Equivariant conformal harmonic maps from the universal cover of a closed Riemann surface into a non-compact symmetric space are useful in ascribing geometric meaning to surface group representations into Lie groups via the non-Abelian Hodge correspondence. After briefly reviewing instances of this for the groups $G = \mathrm{PSL}_2\mathbb{R} \times \mathrm{PSL}_2\mathbb{R}$ and $\mathrm{SL}_3\mathbb{R}$, we discuss $G = \mathrm{Sp}_4\mathbb{R}$, where by work of Collier-Tholozan-Toulisse, the geometric objects are given by equivariant space-like maximal surfaces in the pseudo-hyperbolic space $\mathbb{H}^{2,2}$. We shall describe the limiting behavior of degenerate maximal surfaces, and how these interact when the representations are taken from different connected components of the $\mathrm{Sp}_4\mathbb{R}$-character variety. This is joint work with A. Tamburelli.

Franz Pedit (UMass Amherst, USA)

Title: Higgs bundles, affine spheres, Monge-Ampère equations, and SYZ mirror symmetry

Abstract

We construct special Lagrangian 3-torus fibrations, singular over a trivalent discriminant locus, of Calabi-Yau 3-folds using parabolic non-Abelian Hodge theory. This is an elaboration of an approach first suggested in a paper by Loftin, Yau, and Zaslow.

Tadashi Udagawa (Waseda University, Japan)

Title: Globality of the DPW construction for Smyth potentials in the case of $\mathrm{SU}_{1,1}$

Abstract

In this talk, we construct harmonic maps from Smyth potentials

$$\xi = \frac{1}{\lambda}\begin{pmatrix} 0 & z^{k_0} \\ z^{k_1} & 0 \end{pmatrix} dz, \quad \lambda \in S^1,\ k_0, k_1 \in \mathbb{Z},$$

on $\mathbb{C}^*$ by using the DPW method. The DPW method is a way to construct a CMC-surface, that was developed by J. Dorfmeister, F. Pedit and H. Wu for studying harmonic maps into symmetric spaces. The Gauss map of a constant mean curvature (CMC) surface in $\mathbb{R}^3$ or $\mathbb{R}^{2,1}$ is a harmonic map, thus the DPW method is useful for studying constant mean curvature (CMC)-surfaces. In $\mathbb{R}^{2,1}$ case, the Gauss-Codazzi equation can be regarded as the sinh-Gordon equation

$$u_{t\bar{t}} = e^{2u} - e^{-2u}, \quad u : U \to \mathbb{R},\ U \subset \mathbb{C}.$$

In the DPW method, u is obtained from the Iwasawa factorization of a solution L of $L^{-1}dL = \xi$ in the case of $\mathrm{SU}_{1,1}$. However, the Iwasawa factorization in noncompact case is not always global. Thus, the globality of the Iwasawa factorization is an important problem. In [1], J. Dorfmeister, M. Guest and W. Rossman gave the DPW construction for Smyth potential near z = 0. In this talk, we show that L can be expressed in terms of Bessel functions and from the asymptotic expansion of Bessel functions at large z we solve a Riemann-Hilbert problem to give a global Iwasawa factorization. In this talk, for simplicity, we consider the case $k_0 = 3, k_1 = -1$.

References

[1] J. F. Dorfmeister, M. A. Guest, and W. Rossman, *The tt^* structure of the quantum cohomology of $\mathbb{C}P^1$ from the viewpoint of differential geometry*, Asian J. Math. 14 (2010) 417–438.