Symplectic geometry of the moduli spaces

Indranil Biswas

In the lecture in conference at Takamatsu, which was based in [3] and [4], the symplectic geometry of the moduli spaces of framed logarithmic and parabolic connections, on a Riemann surface with marked points, were described. The first aim in the lecture in Osaka is to describe the symplectic geometry of the moduli spaces of framed meromorphic Higgs bundles on a Riemann surface with marked points; this is based on [1] and [2]. The second aim is to cast the symplectic geometry of moduli spaces of framed logarithmic and parabolic connections and the symplectic geometry of the moduli spaces of framed meromorphic Higgs bundles in the common framework of uniformization of bundles introduced by Tsuchiya, Ueno and Yamada [6]. We show that all these symplectic structures have a common origin when approached from the point of view of uniformization. In fact, they all are given by the Liouville symplectic form on a suitable space associated to the uniformization. This part is based on [5].

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(I. Biswas) School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400005, India
Email address: `indranil@math.tifr.res.in`

Orthogonal ring patterns and discrete surfaces

Alexander Bobenko

Abstract

We introduce orthogonal ring patterns consisting of pairs of concentric circles. They generalize orthogonal circle patterns which can be treated as conformal limit. It is shown that orthogonal ring patterns in euclidean and hyperbolic planes and in a sphere are governed by integrable equations, in particular by the discrete master equation Q_4. We deliver variational principles which are used to prove existence and uniqueness results, and also to compute ring patterns with classical boundary conditions. The later are used to generate discrete cmc surfaces. Relation to minimal surfaces in S^3 and AdS^3 is discussed. Numerous virtual and printed models as well as animation movies will be demonstrated.

(A. Bobenko) Technical University of Berlin, Institute of Mathematics MA 8-4, 10623 Berlin, Germany
Email address: `bobenko@math.tu-berlin.de`

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Higher order generalizations of harmonic maps

Volker Branding

Abstract

Harmonic maps are one of the most famous geometric variational problems for maps between Riemannian manifolds. The harmonic map equation is a second order semilinear elliptic partial differential equation for which many results on questions of existence and qualitative behavior of solutions have been obtained in the literature.

Recently, many researchers got attracted in higher order variants of harmonic maps. In the first part of the talk we will give an overview and present a number of recent results on polyharmonic maps which constitute one possible higher order generalization of harmonic maps. Finally, we will focus on a second higher order version of harmonic maps which was initially proposed by Eells and Sampson in 1964 and present several recent results on the latter.

This is joint work with Stefano Montaldo, Cezar Oniciuc and Andrea Ratto.

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(V. Branding) Faculty of Mathematics, University of Vienna, Oskar-Morgenstern-Platz 1, 1090 Vienna, Austrua

Email address: `volker.branding@univie.ac.at`

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Stability of harmonic maps to symmetric spaces

Fran Burstall

Harmonic maps of Riemannian manifolds are extrema of the energy functional and so it is very natural to try and understand the energy minimisers or, more general, the stable harmonic maps. In this talk, I shall describe some very old work in this area [1, 2] for the case where the target is a Riemannian symmetric space. This will give me an opportunity to pay tribute to Ohnita's fundamental contributions to the topic as well as to point to open questions.

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(F. Burstall) Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, UK
Email address: `feb@bath.ac.uk`

Deformations of spectral curve data

Emma Carberry and Martin U. Schmidt
(Presenter: Emma Carberry)

We combine the construction of deformations of spectral curves based on the Whitham equation [Kr-94] with the theory of deformations of complex spaces and their singularities. Our main result is the construction of a universal deformation of spectral curves for the commonly occuring case of a pair of meromorphic differentials. The universal deformation is parameterised by a finite-dimensional manifold. Our construction applies for example to the spectral curves of periodic solutions of the KdV and sinh-Gordon equation, and doubly-periodic solutions of the KP and the Davey-Stewartson equation. We furthermore modify our construction to produce a universal deformation for a special case of a triple of meromorphic differentials, which applies for example to constant mean curvature tori in three-dimensional space forms.

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(Emma Carberry) School of Mathematics and Statistics, Carslaw Building FO7, The University of Sydney, NSW 2006, Australia
Email address: emma.carberry@sydney.edu.au
webpage: https://www.sydney.edu.au/science/about/our-people/academic-staff/emma-carberry.html

(Martin Ulrich Schmidt) Mathematics Chair III, Universität Mannheim, D-68131 Mannheim, Germany
Email address: schmidt@math.uni-mannheim.de
webpage: https://www.wim.uni-mannheim.de/schmidt/

2020 *Mathematical Subject Classification.* 32G15, 14H15,14H70
Keywords and Phrases. spectral curves, integrable systems, universal deformation, Whitham deformations

Singularities of mean curvature flow

Qing-Ming Cheng

Abstract

Study on singularities of mean curvature flow is one of the most important subjects in study on mean curvature flow. It is well-known that self-shrinkers describe all blow-up at a given singularity of mean curvature flow. Hence, study on self-shrinkers becomes very important. In this talk, we would like to consider complete self-shrinkers of mean curvature flow. We will focus on classifications of n-dimensional complete self-shrinkers of mean curvature flow with the constant length of second fundamental form.

(Q. M. Cheng) Faculty of Science, Fukuoka University, Fukuoka 814-0180, Japan
Email address: `cheng@fukuoka-u.ac.jp`

Fano 3-folds and classification of constantly curved holomorphic 2-spheres of degree 6 in the complex Grassmannian $G(2, 5)$

Quo-Shin Chi

Abstract

Up to now the only known example in the literature of constantly curved holomorphic 2-sphere of degree 6 in the complex $G(2, 5)$ has been the first associated curve of the Veronese curve of degree 4. By exploring the rich interplay between the Riemann sphere and projectively equivalent Fano 3-folds of index 2 and degree 5, we prove, up to the ambient unitary equivalence, that the moduli space of generic (to be precisely defined) such 2-spheres is semialgebraic of dimension 2. All these 2-spheres are verified to have non-parallel second fundamental form.

This is a joint work with Zhenxiao Xie at China University of Mining and Technology, Beijing, and Yan Xu at Nankai University.

(Q. S. Chi) Department of Mathematics and Statistics, Washington University, St. Louis, MO 63130, USA
Email address: chi@wustl.edu

Totally symmetric and and finite uniton type harmonic maps into arbitrary inner symmetric spaces

Josef F. Dorfmeister and Peng Wang
(Presenter: Josef F. Dorfmeister)

Abstract

In this paper, we develop a loop group description (DPW) of harmonic maps $\mathcal{F} : M \to G/K$ of "finite uniton type", from a Riemann surface M into arbitrary inner symmetric spaces (i.e. of compact or of non-compact type). Along this way we also consider harmonic maps for which the monodromy representation is trivial (totally symmetric harmonic maps). Altogether this extends work of Uhlenbeck, Segal, and Burstall-Guest to non-compact inner symmetric spaces. To be more concrete, we prove that every harmonic map of finite uniton type from any Riemann surface into any compact or non-compact inner symmetric space has a normalized potential taking values in some nilpotent Lie subalgebra, as well as a normalized extended frame with initial condition identity. This provides a straightforward way to construct all such harmonic maps. We also illustrate the above results (exclusively) by Willmore surfaces, since the topic of this paper was motivated by the study of Willmore two-spheres in spheres.

(J. F. Dorfmeister) Fakultät für Mathematik, TU-München, Boltzmannstr.3, D-85747, Garching, Germany
Email address: dorfmeis@gmail.de

(P. Wang) College of Mathematics & Informatics, FJKLMAA, Fujian Normal University, Qishan Campus, Fuzhou 350117, P. R. China
Email address: pengwang@fjnu.edu.cn

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Keywords and Phrases. harmonic maps of finite uniton type; non-compact inner symmetric spaces; normalized potential; Willmore surfaces.

On the existence of holomorphic curves in compact quotients of $SL(2, \mathbb{C})$

Lynn Heller

Abstract

In my talk I will report on recent joint work with I. Biswas, S. Dumitrescu and S. Heller showing the existence holomorphic maps from a compact Riemann surface of genus $g > 1$ into a quotient of $SL(2, \mathbb{C})$ modulo a cocompact lattice which is generically injective. This gives an affirmative answer to a question raised by Huckleberry and Winkelmann and by Ghys. The proof uses The proof uses ideas from harmonic maps into the hyperbolic 3-space, WKB analysis, and the grafting of real projective structures.

(L. Heller) Beijing Institute for Mathematical Sciences and Applications, Yanqi Island, Huairou District, Beijing 101408
Email address: `lynn@bimsa.cn`

New minimal Lagrangian surfaces in $\mathbb{C}P^2$

Sebastian Heller

Abstract

A surface $f\colon \Sigma \to \mathbb{C}P^2$ is called minimal Lagrangian if it is both, a minimal surface with respect to the Fubini study metric and Lagrangian with respect to the Kähler form. Besides the real projective plane and plenty of minimal Lagrangian tori, the only known compact examples have been constructed by Haskins and Kapouleas [2]. In this talk, I will explain the construction of new compact minimal Lagrangian surfaces using loop group factorization methods. Note that there are no compact embedded oriented Lagrangian surfaces in $\mathbb{C}P^2$, and the newly constructed examples in $\mathbb{C}P^2$ are only immersed.

(Branched) minimal Lagrangian surfaces from a Riemann surface Σ can be obtained from holomorphic families of flat $\mathrm{SL}(3,\mathbb{C})$-connections

$$\lambda \in \mathbb{C}^* \mapsto \lambda^{-1}\Phi^{1,0} + \nabla + \lambda\Phi^{0,1}$$

which satisfy the following properties:

1. $\Phi^{1,0}$ is an endomorphism-valued $(1,0)$-form on Σ (harmonicity);
2. ∇^λ is unitary for $\lambda \in S^1$ and trivial for $\lambda = 1$ (intrinsic and extrinsic closing);
3. ∇^λ and $\nabla^{\epsilon\lambda}$ are gauge equivalent by a λ-independent gauge, where $\epsilon = \exp(\frac{2\pi i}{3})$ (conformality);
4. ∇^λ and $\nabla^{-\lambda}$ are dual to each other (Lagrangian).

In my talk, I explain how to construct families of flat connections satisfying these properties on certain Riemann surfaces of large genus. Furthermore, I will show that the corresponding minimal Lagrangian surfaces are immersions, and derive some of their geometric properties. For example, I show that they lift to special Legendrian surfaces (SL) in $\mathbb{S}^5 \to \mathbb{C}P^2$, giving the first proof of existence of compact SLs of arbitrarily large even genus. If time permits, I will explain the relationship between our approach and the construction of local minimal Lagrangians via twisted potentials [1].This talked is based on joint work [3] with Charles Ouyang and Franz Pedit.

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(S. Heller) Beijing Institute for Mathematical Sciences and Applications, Yanqi Island, Huairou District, Beijing 101408

Email address: `sheller@bimsa.cn`

Symmetry breaking

Udo Hertrich-Jeromin

Abstract

Symmetry plays a key role in geometry and physics alike: in both disciplines the notion has been studied and was formalized around the turn of the 20th century, most notably by F Klein (1872 and 1893) in geometry [3] and by P Curie (1894) in physics [1]. In physics, these developments were accompanied by the observation and the study of "symmetry breaking"; in geometry, this notion has received little to no attention. However, also in geometry, there are phenomena that display a sort of "symmetry breaking", in a way that was described by F Klein in his Erlangen programme and that features some similarities to what is considered "symmetry breaking" in physics.

We shall investigate some of these phenomena in geometry, some related to integrable reductions of the corresponding differential equations; discuss similarlties and differences to the notion of symmetry breaking in physics; and attempt a working definition of "symmetry breaking" in geometry based on Klein's work.

This presentation is based on long-term joint work with numerous colleagues and, in particular, on a joint paper with A Fuchs and M Pember [2].

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(U Hertrich-Jeromin) Institute of discrete mathematics and geometry, University of Technology Vienna, Wiedner Hauptstrasse 8-10/104, 1040 Wien (Austria)
Email address: `udo.hertrich-jeromin@tuwien.ac.at`

Genaration of aesthetic shapes by integrable geometry

Kenji Kajiwara

In this talk, we consider a class of plane curves called log-aesthetic curves (LAC) and their generalizations which have been developed in industrial design as the curves obtained by extracting the common properties among thousands of curves that car designers regard as aesthetic. We consider these curves in the framework of similarity geometry and characterize them as invariant curves under the integrable deformation of plane curves governed by the Burgers equation. We propose a variational principle for these curves, leading to the stationary Burgers equation as the Euler-Lagrange equation.

We then extend the LAC to space curves by considering the integrable deformation of space curves under similarity geometry. The deformation is governed by the coupled system of the modified KdV equation satisfied by the similarity torsion and a linear equation satisfied by the curvature radius. The curves also allow the deformation governed by the coupled system of the sine-Gordon equation and associated linear equation. The space curves corresponding to the travelling wave solutions of those equations would give generalization of LAC to space curves.

We also consider the surface constructed by the family of curves obtained by the integrable deformation of such curves. A special class of surfaces corresponding to the constant similarity torsion yields quadratic surfaces (ellipsoid, one/two-sheeted hyperboloids and paraboloid) and their deformations, which may be regarded as a generalization of LAC to surface.

We discuss the construction of such curves and surfaces together with their mathematical properties, including integration scheme of the frame by symmetries, and present various examples of curves and surfaces.

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(K. Kajiwara) Institute of Mathematics for Industry, Kyushu University, 744 Motooka, Nishi-ku, Fukuoka 819-0395, Japan.
Email address: `kaji@imi.kyushu-u.ac.jp`

This work was supported by JST CREST JPMJCR1911 and JSPS Kakenhi 21K03329.

Eigenspaces and minimal surfaces

Robert Kusner

Abstract

We develop sharp bounds for the dimensions of first Steklov (or Laplacian) eigenspaces on surfaces, using this to study the geometry of free boundary (or closed) minimal surfaces in the round ball (or sphere).

(R. Kusner) Department of Mathematics, University of Massachusetts, Amherst, MA 01003, USA

Email address: kusner@umass.edu

Periodic Darboux transforms

Katrin Leschke

In classical differential geometry, geometric transformations have been used to create new curves and surfaces from simple ones: the aim is to solve the underlying defining compatibility equations of curve or surface classes by finding solutions to a simpler system of differential equations arising from the transforms. Classically, the main concern was a local theory. In modern theory, global questions have led to a renewed interest in classical tranformations. For example, in the case of a torus, the investigation of closing conditions for Darboux transforms naturally leads to the notion of the spectral curve of the torus.

In this talk we discuss closing conditions for smooth and discrete polarised curves, isothermic surfaces and CMC surfaces. In particular, we obtain new explicit periodic discrete polarised curves, new discrete isothermic tori and new explicit smooth CMC cylinder.

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(K. Leschke) School of Computing and Mathematical Sciences, University of Leicester, Leicester LE17RH, United Kingdom
Email address: `k.leschke@leicester.ac.uk`

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Construction of minimal Lagrangian surfaces in $\mathbb{C}P^2$ via integrable system

Hui Ma

In this talk I will discuss the construction of minimal Lagrangian surfaces in the complex projective plane by loop group method. This is based on the joint work with Josef Dorfmeister and Shimpei Kobayashi.

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(H. Ma) Department of mathematical Sciences, Tsinghua University, Beijing 100084, P.R. China
Email address: `ma-h@tsinghua.edu.cn`

Review of my research

Reiko Miyaoka

I give a survey talk, looking back on my research including a joint work with Y. Ohnita.

First, I mention my result on minimal surfaces [1] motivated by Lawson's work related to Calabi's rigidity theorem. This is because Calabi is 100 years old this year. Much later, I found the subject to be related to the Affine Toda equation [6].

Next, I talk about Dupin hypersurfaces and the Lie sphere geometry, started from U. Pinkall's thesis. The most important result of mine is the discovery of the "Lie curvature" [3], that is a key tool to resolve the Cecil-Ryan conjecture [2], [4]. I continue to study the Lie contact structure, and solved H. Sato's problem using the Cartan-Tanaka connection [5].

Finally, I've focussed on the theory of isoparametric hypersurfaces [7], and using them as a subject, we got some results on Floer homology with Ohnita et al. [8].

When time allows, I'll show you a few pictures for fun.

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(R. Miyaoka) Tohoku University, Mathematical Institute, 6-3, Aoba-ku, Sendai, 980-8578, Japan
Email address: `r-miyaok@tohoku.ac.jp`

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Transforms of minimal surfaces

Katsuhiro Moriya

Harmonic maps are one of Ohnita's research subjects. That made his interest in integrable systems. A harmonic map is minimal if it is an isometric immersion. In this talk, I shall define transforms of minimal surfaces.

(K. Moriya) Mathematical Science Laboratoy, Department of Material Science, University of Hyogo, 2167, Shosha, Himeji, Hyogo 671-2280, Japan
Email address: m905k019@guh.u-hyogo.ac.jp

The author was partly supported byJSPS KAKENHI Grant Number JP22K03293.

The Gaiotto correspondence, quantization, and connections defined over $\overline{\mathbb{Q}}$

Motohico Mulase

1 Abstract

The talk is aimed at presenting a simple mathematical description of the *conformal limit* construction of Davide Gaiotto [3, 4], and its potential implications in problems on $\overline{\mathbb{Q}}$-geometry and mirror symmetry. The talk is based on my ongoing collaboration with Alex Cruz Morales, Olivia Dumitrescu, and Jon Erickson.

The Gaiotto construction assigns an *oper*, a connection on a Riemann surface C with a complex variation of Hodge structure, to a Higgs bundle on C equipped with an $SL(2)$-structure. This process can be thought of as a *quantization* of a Hitchin spectral curve to a Schrödinger operator [1, 2].

Starting from the classical Riemann surface theory, such as projective structures, the Schwarzian derivative, holomorphic quadratic differentials, and the constant curvature metric on a compact Riemann surface of positive genus, we can see the interplay between Kostant's *Three-Dimensional Subgroups* in a simple complex algebraic group G, quantization processes, and variations of Hodge structure, via analyzing the Gaiotto construction.

This construction also defines a biholomorphic correspondence between a complex Lagrangian in the Dolbeault moduli space of G-Higgs bundles and a complex Lagrangian in the de Rham moduli space of G-connections. Through the *hyperkähler rotation* and *Langlands duality*, the Gaiotto correspondence is expected to play a key role in understanding the mirror symmetry of these moduli spaces.

The talk is dedicated to Professor Yoshihiro Ohnita at his retirement.

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Department of Mathematics, University of California, Davis
One Shields Avenue, Davis, California 95616, U.S.A.
Email address: `mulase@math.ucdavis.edu`

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Global geometry of q-Painlevé equations

Yousuke Ohyama

We study connection problem of q-Painlevé equations. As the same as classical Painlevé differential equations, q-Painlevé equations are also represented by a connection preserving deformation of a q-linear difference equation. And a q-analogue of the Riemann-Hilbert correspondence plays an important role to study q-Painlevé functions.

We review connection problems of q-linear difference equations at first. Then we study q-Riemann-Hilbert problem on the q-Painlevé VI equation [1, 2]. We show that the space of connection is a two dimensional algebraic surface, an open subvariety of the Segre surface.

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(Y. Ohyama) Department of Mathematicall Sciences, Tokushima University, 2-1 Minamijyousanjima-cho, Tokushima 770-8506, Japan
Email address: `ohyama@tokushima-u.ac.jp`

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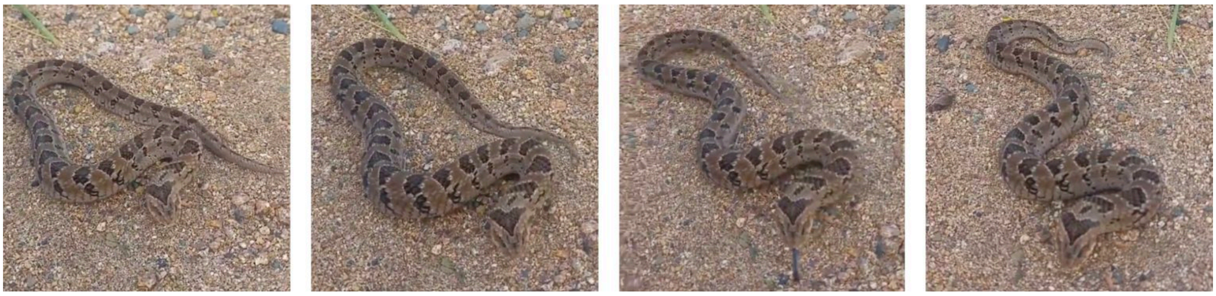
Snakes, falling cats and fiber bundles

Oliver Gross, Felix Knöppel, Marcel Padilla, Ulrich Pinkall, Peter Schröder
Presenter: Ulrich Pinkall

1 Locomotion of Snakes



Imagine a snake of length L moving on the plane $\mathbb{R}^2$ in such a way that all parts of its body stay in contact with the ground. Then at each time t the position of the snake can be modeled by an element $\gamma(t)$ of the space

$$\mathcal{M} = \left\{ \gamma \in C^\infty([0, L], \mathbb{R}^2) \;\middle|\; |\gamma'| = 1 \right\}$$

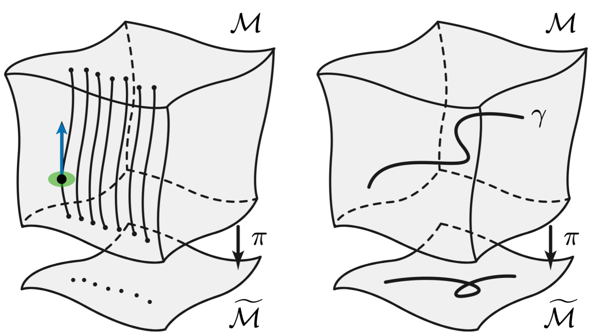
of all arclength-parametrized plane curves of length L. The group SE(2) of orientation-preserving Euclidean motions of $\mathbb{R}^2$ acts on $\mathcal{M}$. At each time t the *shape* of the snake is given by the orbit

$$S_t = \{ g \circ \gamma(t) \mid g \in \mathrm{SE}(2) \}$$

under this group action. This shape can be described by a certain curvature function, so the space of all possible shapes is given by

$$\widetilde{\mathcal{M}} = \mathcal{M}/_{\mathrm{SE}(2)} \simeq C^\infty[0, L].$$

The projection map $\pi \colon \mathcal{M} \to \widetilde{\mathcal{M}}$ makes $\mathcal{M}$ into the total space of a principal fiber bundle with structure group SE(2) over the base manifold $\widetilde{\mathcal{M}}$.



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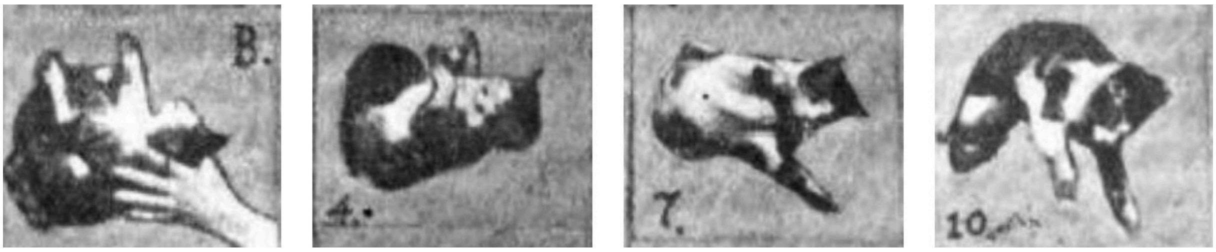
The snake controls only the sequence $t \mapsto S_t \in \widetilde{\mathcal{M}}$ of shapes. What is the resulting sequence $t \mapsto \gamma(t) \in \mathcal{M}$ of snake positions? The mathematical model we propose is that $t \mapsto \gamma(t)$ is a horizontal lift (with respect to a certain connection on the bundle) of the given curve $t \mapsto S_t$ in the base manifold. At each point $\gamma \in \mathcal{M}$ the horizontal space of this connection is the orthogonal complement of the tangent space to the fiber through γ with respect to a certain Riemannian metric on $\mathcal{M}$. This metric assigns to each velocity vector $\dot\gamma \in T_\gamma\mathcal{M}$ the corresponding energy dissipation $\frac{1}{2}\langle\dot\gamma, \dot\gamma\rangle$ due to friction with the ground. Putting aside the question how to compute this metric, we apply Helmholtz' *principle of least dissipation*: Among all possible lifts of the given curve in the base manifold the corresponding lift γ to the total space minimizes dissipation, i.e. the Riemannian energy of γ. In other words: the resulting snake motion γ is a horizontal lift of the given curve S in shape space.

2 Falling Cats

Imagine a cat initially at rest in a position with its feet pointing upward. After being released, the cat starts changing its shape and finally achieves that its feet point downward. We can ignore gravity because with respect to the center of mass the resulting motion will be the same as in the corresponding situation with no gravity.

This situation resembles the one of the snake: now the group SE(3) acts on the space $\mathcal{M}$ of cat positions and the shape space $\widetilde{\mathcal{M}}$ is the corresponding quotient space of $\mathcal{M}$. This time, the Riemannian geometry of $\mathcal{M}$ is given by the kinetic energy of the cat and the variational principle we apply is the *principle of least action*.



The same mathematical model also applies to shape-changing bodies under water. The only difference is that here we also have to take into account the kinetic energy of the motion of the surrounding fluid (enforced by the shape change of the body).

The above mathematical setup can be turned into a numerical algorithm that can be efficiently used in the context of Computer Graphics.

(U. Pinkall) Technical University of Berlin MA 3-2, Str. d. 17. Juni 136, 10623 Berlin, Germany
Email address: `upinkall@gmail.com`

Canonical coordinates of Moduli spaces of Connections and Higgs bundles, Isomonodromic deformations and differential equations of Painlevé type

Masa-Hiko Saito

1 Moduli Space of Connections and the Riemann-Hilbert correspondence

The moduli spaces of singular connections and singular Higgs bundles on curves are constructed by geometric invariant theory. Using stable parabolic connections and Higgs bundles, we can construct moduli spaces M_{DR} and M_H of these geometric objects as smooth quasi projective algebraic varieties. Moreover, we will denote by M_{mon} the moduli of monodromy representation of the fundamental group, or generalized monodromy such as Stokes data, etc. Note that Classical Painlevé equations, classified into 8 types, correspond to 10 types of generalized data and the equations for M_{mond} are given in [PS]. Under a suitable choice of parameter, we can define the Riemann-Hilbert correspondence

$$\mathsf{RH} : M_{DR} \longrightarrow M_{mon}$$

Usually, the left-hand side forms a family of moduli spaces of connections parametrized by time variables coming from the datum of regular or irregular singular points. Note that the right-hand side does not depend on time variables, at least locally.

Theorem 1.1 *([IIS1], [Ina], [IS1], [IS2]) We assume that all singularities of connections are regular or unramified irregular. Fixing a time variable,* RH *is a proper surjective bimeromorphic holomorphic map.*

In the generic case, RH is an analytic isomorphism. If M_{mon} has singularities, RH becomes an analytical resolution of singularities. Pulling back the constant sections on the right-hand side (monodromy side) by the Reimann-Hilbert correspondence, we can define the set of flows on the family of moduli spaces of connections on the right-hand side. This is called *Isomonodromic Flows* defined on the family of moduli space of connections. We can define the differential equations describing these isomonodromic flows, which we call isomonodromic differential equations. By a theorem due to Inaba, Iwasaki and Saito and other people, we can show the following.

Theorem 1.2 *([IIS1], [Ina], [IS1], [IS2]) The isomonodromic differential equations coming from the moduli space of connections with regular or unramified singularities satisfy* Painlevé propety*, that is, all of the movable singularities of the solutions of the isomonodromic differential equations are poles.*

We remark here that it is essential to construct the moduli spaces in the category of algebraic varieties.

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2 Symplectc structures and Canonical coordinates for moduli spaces

It is known that all moduli spaces M_{DR}, M_H, M_{mon} admit natural symplectic structures ([IIS1], [Ina], [IS1], [IS2]). One can see that RH preserves these symplectic structures. Moreover on a Zarisiki dense open set of moduli spaces of M_{DR}, M_H, we can introduce canonical coodinates of M_{DR}, M_H by using apparent singularities and their duals ([SSz]). We will explain these facts in new settings and the relation to the works of Iwasaki, Dubrovin-Mazzocco and Krichever.

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(Masa-Hiko, Saito), Faculty of Bussiness Administrations, Kobe Gakuin University, 1-1-3, Minatojima, Chuo-ku, Kobe, 650-8586, Japan.
Email address: `mhsaito@ba.kobegakuin.ac.jp`

On two questions of I. M. James

Zizhou Tang

65 years ago, I. M. James raised two fundamental questions about octonionic Stiefel spaces. The prime objective of this talk is to give partial answers to both of them. This is based on the joint work with Chao Qian and Wenjiao Yan.

(Z. Tang) Chern Institute of Mathematics, Nankai University, Tianjin, 300071, China
Email address: zztang@nankai.edu.cn

Characterizations of Ricci flat metrics and Lagrangian submanifolds in terms of the variational problem

Tetsuya Taniguchi

1 Abstract

Given the pair (P, η) of $(0, 2)$ tensors, where η defines a volume element, we consider a new variational problem varying η only. We then have Einstein metrics and slant immersions as critical points of the 1st variation. We may characterize Ricci flat metrics and Lagrangian submanifolds as stable critical points of our variational problem.

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(T. Taniguchi) Nihon University School of Medicine, Liberal Arts, 30-1 Oyaguchi Kamicho, Itabashi-ku, Tokyo, 173-8610, Japan
Email address: `taniguchi.tetsuya@nihon-u.ac.jp`

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Solving the sine-Gordon equation and its discretization

Seiichi Udagawa

Abstract

The sine-Gordon equation has played an important role in the study of integrable systems and there are many works related to it.
1. Its (see [2]) obtained the solution of SG-equation in terms of Riemann θ-function.
2. The sine-Gordon equation is discretized by Ryogo Hirota (see [4]). The solution in terms of the Riemann θ-function was obtained by Bobenko-Pinkall (see [1]).
3. On the other hand, a semi-discrete version can be considered in which position coordinates are discretized with one independent variable as a continuous time parameter. This is recently attracting attention as an equation that describes the equation of motion of the Kaleidocycle (see [5], [8]).

In this talk, 1. First, we obtain the elliptic function solution of the sine-Gordon equation and confirm that it can be expressed in the form of the solution of the expression of the Riemann θ-function by Its.
2. Next, we obtain the elliptic function solution for the semi-discrete sine-Gordon equation, and again describe the relationship with the expression by the Riemann θ-function.
3. Moreover, we obtain the elliptic function solution for the discrete sine-Gordon equation and describe its relationship with the expression by the Riemann θ-function by Bobenko-Pinkall.
4. The ultimate goal is to describe the motion of the Kaleidcycle using elliptic functions. Recently, Shigetomi (Kyushu University) described it in terms of Jacobi θ-functions.

This is a joint work with J. Inoguchi and K. Kajiwara.

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(S. Udagawa) Nihon University, Itabashi 173-0032, Tokyo, Japan
Email address: `udagawa.seiichi@nihon-u.ac.jp`

Ray Structures on Teichmüller Space

Huiping Pan and Michael Wolf
(Presenter: Mike Wolf)

Abstract

There are several classical ray structures on Teichmüller space, including the Teichmüller geodesics and the Thurston geodesics. The former refer to the conformal structure of a Riemann surface and the latter to the hyperbolic structure. There are also harmonic map rays, which depend on a fixed conformal domain and varying hyperbolic targets: how do these fit into the family of ray structures. We depict harmonic map ray structures on Teichmüller space as a geometric transition between Teichmüller ray structures and Thurston geodesic ray structures.

As a consequence, while there may be many Thurston metric geodesics between a pair of points in Teichmüller space, we find that by imposing an additional energy minimization constraint on the geodesics, thought of as limits of harmonic map rays, we select a unique Thurston geodesic through those points. Extending the target surface to the Thurston boundary yields, for each point Y in Teichmüller space, an "exponential map" of rays from that point Y onto Teichmüller space with visual boundary the Thurston boundary of Teichmüller space.

(M. Wolf) School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia, USA
Email address: `mwolf40@gatech.edu`

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Symplectic structures on almost abelian Lie algebras

Luis Pedro Castellanos Moscoso

Abstract

We are interested in the classification or finding conditions for the existence of left-invariant symplectic structures on Lie groups. Only some classifications are known, specially in low dimensions. We approach this problem by studying the "moduli space of left-invariant nondegenerate 2-forms", which is a certain orbit space in the set of all non-degenerate 2-forms on a Lie algebra. In this talk we consider this problem in the case of almost abelian Lie algebras, that is Lie algebras that contain a codimension 1 abelian subalgebra. Almost abelian Lie algebras can be described in terms of linear operators. The isomorphism classes of almost abelian Lie algebras are related to the similarity classes of these linear operators. Then, we can show that, in this setting, the problem of existence and classification of left-invariant symplectic structures can be reduced to a known matrix equation involving the Jordan normal form of a matrix and a corresponding equivalence relation, respectively. We study the solution of these equations for several particular examples.

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(Luis Pedro Castellanos Moscoso) Osaka Central Advanced Mathematical Institute, 3-3-138, Sugimoto, Sumiyoshi-ku, Osaka, 558-8585, Japan
Email address: caste3.1416@gmail.com

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Darboux transformations in the Lorentz-Minkowski plane

Masaya Hara

This talk concerns Lorentz-Möbius geometry, and Darboux transformations in that context. We construct the basic structure of Lorentz-Möbius geometry by using the notion of the generalized point sphere complex, and we show some unique properties of Darboux transformations in the Lorentz-Minkowski plane $\mathbb{R}^{1,1}$, especially with regard to singularities and unboundedness. Additionally, we describe suggestive Darboux transformations of type-changing curves in $\mathbb{R}^{1,1}$.

1 Darboux transformations

Let $\mathbb{R}^{1,1}$ denote a 2-dimensional pseudo-Riemannian space with signature $(+-)$ and inner product $\langle \cdot, \cdot \rangle_{1,1}$, that is, for $x, y \in \mathbb{R}^{1,1}$,

$$\langle x, y \rangle_{1,1} = \langle (x_1, x_2)^t, (y_1, y_2)^t \rangle_{1,1} = x_1 y_1 - x_2 y_2,$$

and we use $|\cdot|^2$ to denote the inner product of a vector with itself, i.e. the length squared in $\mathbb{R}^{1,1}$.

We can define the Darboux transformations of curves in $\mathbb{R}^{1,1}$ by the Riccati-type equation as follows:

Definition 1.1 ([1, 2]). Let $x(t) : \left(I, \frac{dt^2}{m}\right) \to \mathbb{R}^{1,1}$ be a smooth curve polarized by $m(t) : I \to \mathbb{R}\backslash\{0\}$. For some nonzero constant $\mu \in \mathbb{R}\backslash\{0\}$, a curve $\hat{x}(t)$ defined by

$$\hat{x}' = \mu |\hat{x} - x|^2 x^{*\prime} - 2\mu \langle \hat{x} - x, x^{*\prime} \rangle_{1,1} (\hat{x} - x)$$

is called a *Darboux transformation with parameter* μ of $x(t)$, where $x^{*\prime} = -\dfrac{x'}{m|x'|^2}$.

In Figure 1, the blue line is the initial line and an orange curve is the Darboux transformation with the black point as initial condition. The spectral parameter μ is positive (resp. negative) in the left picture (resp. right picture).

2 Properties

Let $\hat{x}$ be a Darboux transformation of a spacelike curve x. We show some properties of such Darboux transformations:

- $\hat{x}$ can be lightlike at some points and the common circle congruence S becomes degenerate there.
- If x is singularity free, $\hat{x}$ is also singularity free.
- If x and $\hat{x}$ have the common arc-length polarization m, $\hat{x}$ is unbounded if and only if S is degenerate.
- x can be unbounded with finite limiting curvature.

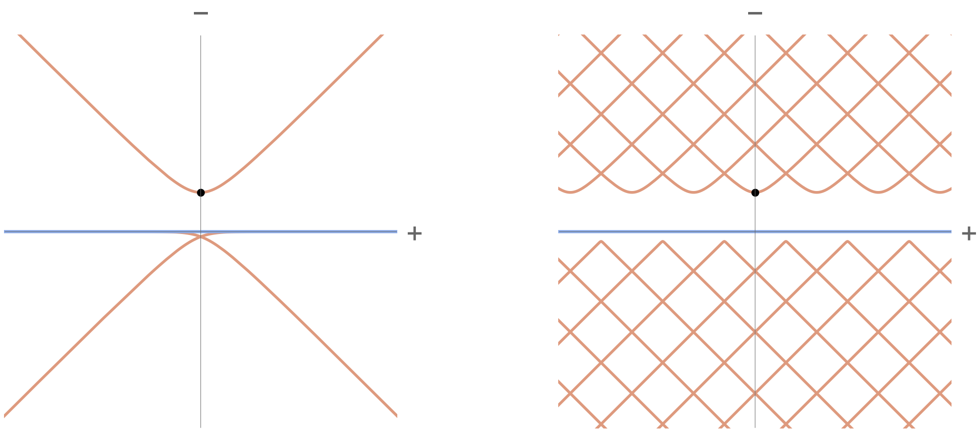

Figure 1: Examples of Darboux transformations of a spacelike line.

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(M. Hara) Department of Mathematics, Graduate School of Science, Kobe University, 1-1, Rokkodai-cho, Nada-ku, Kobe 657-8501, Japan
Email address: `mhara@math.kobe-u.ac.jp`

On some Lie-theoretic solutions of the tt*-Toda equations

Yudai Hateruma

The tt* equations are an integrable system of partial differential equations. They were introduced by Cecotti and Vafa in their work on $N = 2$ supersymmetric quantum field theories. From the physical point of view, Cecotti and Vafa made several conjectures regarding solutions of these equations, in particular the solutions, whose Stokes matrices have integer entires. Guest, Its and Lin studied the tt* equations of 'Toda type' for the group $SL(n + 1, \mathbb{C})$. They described all global solutions and some of their properties. From the Lie theoretic point of view, the set of the global solutions is in one-to-one correspondence with the Fundamental Weyl Alcove. However describing properties of the 'integer Stokes solutions' is still difficult. In this talk, we describe the following two results of these solutions. First, we describe all integer Stokes solutions in a certain one dimensional subspace of the Fundamental Weyl Alcove. Second, we compare the results of Guest-Its-Lin and the predictions of Cecotti-Vafa. We find that, in the case of the tt*-Toda equations, these two methods agree, at least for low values of n. This is joint work with Yoshiki Kaneko.

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(Y. Hateruma) Department of Pure and Applied Mathematics, Faculty of Science and Engineering, Waseda University, 3-4-1 Okubo, Shinjuku, Tokyo 169-8555, Japan

Email address: `wuyh2.17-1@asagi.waseda.jp`

On the truss structures in architectural design based on discrete isothermic surfaces

Yoshiki Jikumaru, Kazuki Hayashi, Kentaro Hayakawa,
Yohei Yokosuka, and Kenji Kajiwara
(Presenter: Yoshiki Jikumaru)

Abstract. A classical truss structure discovered by A. G. M. Michell [3] is known as a structure which transmits loads efficiently with minimum possible members. In this talk, we discuss a truss structure analogous to the Michell truss constructed from discrete isothermic surfaces. We show that our proposed structure has not only nice mechanical properties and constructibility, but also a deep connection with log-aesthetic curves.

1 Mechanical properties of discrete isothermic surfaces [1, 5]

Let us consider a map $\boldsymbol{r} : \mathbb{Z}^2 \to \mathbb{R}^3$, where $\mathbb{Z}^2$ is a planar lattice and $\mathbb{R}^3$ is the 3-dimensional Euclidean space. For a vertex (n, m) in $\mathbb{Z}^2$, denote $\boldsymbol{r} = \boldsymbol{r}(n, m)$, $\boldsymbol{r}_{(1)} = \boldsymbol{r}(n+1, m)$, $\boldsymbol{r}_{(2)} = \boldsymbol{r}(n, m+1)$, and $\boldsymbol{r}_{(12)} = \boldsymbol{r}(n+1, m+1)$. In the following, we assume all the quadrilaterals $[\boldsymbol{r}, \boldsymbol{r}_{(1)}, \boldsymbol{r}_{(12)}, \boldsymbol{r}_{(2)}]$ are planar and concircular. Such a map $\boldsymbol{r}$ is often called a *discrete curvature net* or *circular net*. We also assume that the internal forces $\boldsymbol{F}_1$, $\boldsymbol{F}_2$, $\boldsymbol{F}_{1(1)}$ and $\boldsymbol{F}_{2(2)}$ act tangentially at the midpoints of the edges, see Fig. 1.

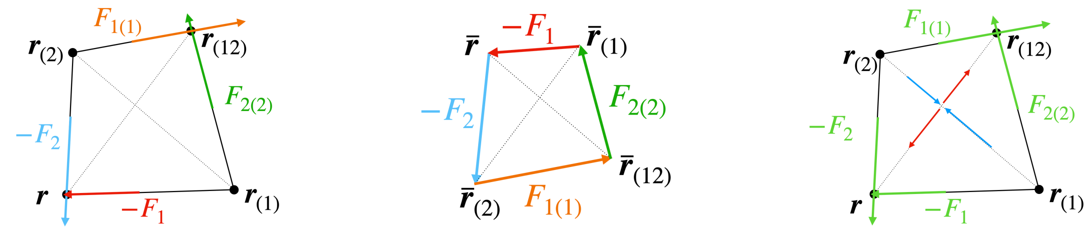

Figure 1: (a): Internal forces act tangentially on a quadrilateral. (b): The force balancing condition induces another (closed) quadrilateral. (c): Axial forces acting on the diagonals.

Theorem 1.1 (Schief [5], Theorem 3.2)**.** *Assume that purely tangential internal forces act at the midpoints of the edge of a discrete curvature net* $\boldsymbol{r}$*. Then* $\boldsymbol{r}$ *is in equilibrium if and only if* $\boldsymbol{r}$ *constitutes a discrete isothermic surface, and the internal forces are encoded in the discrete Christoffel transform* $\overline{\boldsymbol{r}}$*.*

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2 Michell truss-like structures [3, 2]

Under the assumption of Theorem 1.1, by replacing the quadrilaterals with their diagonals, we obtain a truss structure in equilibrium transmitting only by their axial forces, see Fig. 1 (c). In particular, we consider a planar discrete isothermic surface which is called an *(integrable) discrete holomorphic function* or *discrete conformal map* [1, 4].

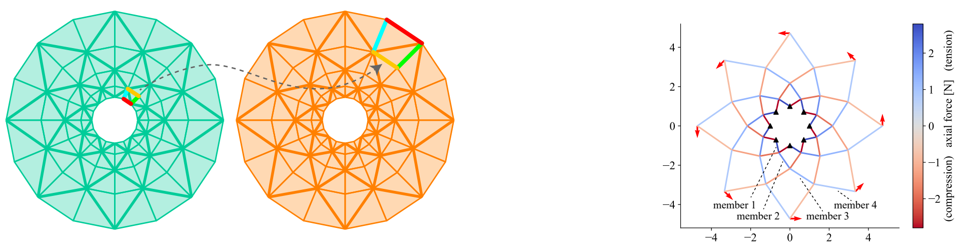
Let us consider a map $\boldsymbol{r}(n,m) = \exp(\rho n + \sqrt{-1}\kappa m)$, where ρ and κ are constant. A direct calculation shows that $\boldsymbol{r}$ is discrete holomorphic if and only if

$$\sinh^2\frac{\rho}{2} = \sin^2\frac{\kappa}{2} \iff \cosh\rho + \cos\kappa = 2. \tag{2.1}$$

holds. If ρ and κ satisfy this condition, then we call $\boldsymbol{r}$ *discrete exponential function.* Moreover, the Christoffel transform $\overline{\boldsymbol{r}}$ of $\boldsymbol{r}$ is given as follows:

$$\overline{\boldsymbol{r}}(n,m) = -\frac{1}{4\sinh^2(\rho/2)}\exp(-\rho n + \sqrt{-1}\kappa m). \tag{2.2}$$

We can check that the structure is certainly minimizing the sum of the products of



(a) (b)

Figure 2: (a): A discrete exponential function and its Christoffel transform. The axial forces acting on the inner member of the structure is greater. (b): The result of structural analysis. The loading is applied in the direction of twisting the circumference.

absolute value of axial force and the member length numerically, which satisfies Michell's original problem [3] as mentioned. Through the inverse of the stereographic projection, we obtain a truss structure on the unit sphere as shown in Fig. 3.

We finally remark that discrete curves obtained from the diagonals of the discrete exponential function are integrable discretization of log-aesthetic curves. The notion of log-aesthetic curves (LAC) has been originally proposed by extracting the common properties from thousands of plane curves that car designers regard as aesthetic. It was shown that LAC can be characterized by the invariant curves of integrable deformation of plane curves in similarity geometry [2]. An integrable discretization of LAC (dLAC) is proposed by Inoguchi et al. [2] as follows. A sequence of points in the plane $\gamma_n \in \mathbb{C}$ $(n \in \mathbb{Z})$ is called a *discrete curve* and the angle $\kappa_n = \angle(\gamma_n - \gamma_{n-1}, \gamma_{n+1} - \gamma_n)$ is called the *turning angle.* We put $q_n := |\gamma_{n+1} - \gamma_n|$ and $u_n := q_{n+1}/q_n$. u_n is called the *discrete similarity curvature.*

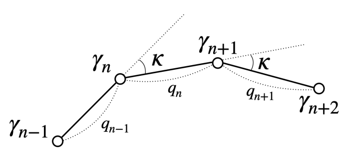
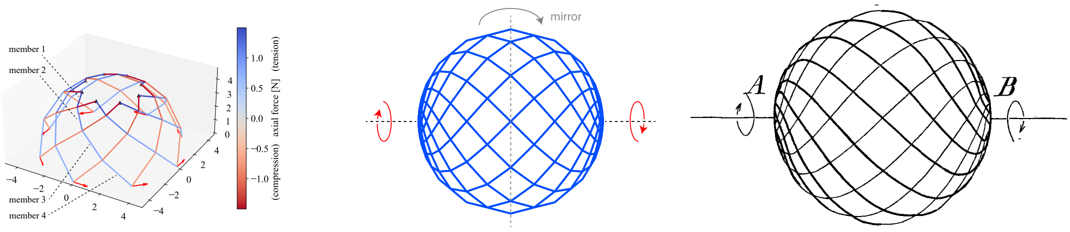


Figure 4: Notations for a discrete curve γ_n

Definition 2.1 ([2])**.** *A discrete curve $\gamma_n \in \mathbb{C}$ is called a* discrete log-aesthetic curve (dLAC) of slope α *if the following conditions are satisfied:*



(a) (b) (c)

Figure 3: (a): A truss structure on the unit sphere by the inverse of the streographic projection. The loading is applied in the direction of twisting the equator. (b): A structure attached along the equator. (c): A structure described in [3].

1. *γ_n has constant turning angle $\kappa_n = \kappa$.*
2. *u_n satisfies the equation*

$$u_n = \left(1 + \frac{(\alpha-1)\lambda\kappa}{(\alpha-1)\lambda\kappa n + 1}\right)^{\frac{1}{\alpha-1}} \text{ if } \alpha \neq 1, \quad u_n = e^{\lambda\kappa} \text{ if } \alpha = 1, \tag{2.3}$$

where λ is a constant.

The discrete curve γ_n obtained from the diagonals of a discrete exponential function $\boldsymbol{r}$ can be written in the form

$$\gamma_n = \boldsymbol{r}_0 \exp((\rho \pm \sqrt{-1}\kappa)n), \quad n \in \mathbb{Z}, \tag{2.4}$$

where $\boldsymbol{r}_0 \in \mathbb{C}$ is a fixed point. Then we can show the following result:

Proposition 2.2. *The discrete curve γ_n in the form* (2.4) *has a constant turning angle κ, and satisfies $u_n = e^{\rho}$. In other words, γ_n is dLAC of slope* 1.

An LAC of slope 1 corresponds to a logarithmic spiral in the smooth setting. Therefore a discrete curve defined in the form (2.4) can be regarded as a discrete logarithmic spiral.

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(Y. Jikumaru) Institute of Mathematics for Industry, Kyushu University, 744, Motooka, Nishi-ku, Fukuoka, 819-0395, Japan
Email address: `y-jikumaru@imi.kyushu-u.ac.jp`

A nonclassical algebraic solution of a 3-variable irregular Garnier system

Arata Komyo

This talk is based on [4]. So a nonclassical algebraic solution of a 3-variable irregular Garnier system is constructed. Diarra–Loray have studied classification of algebraic solutions of irregular Garnier systems ([1]). There are two type of the algebraic solutions: classical type and pull-back type. They have shown that there are exactly three nonclassical algebraic solutions for N-variables irregular Garnier systems with $N > 1$. Explicit forms of two of the three solutions are already given. However, an explicit form of the remaining algebraic solution is not given in [1]. This is because the corresponding explicit irregular Garnier system had not been known. On the other hand, there are studies of explicit description of isomonodromic deformations of second order linear differential equations with irregular singularities by using apparent singularities ([2, 3]). By applying these studies, we construct an explicit form of the remaining algebraic solution.

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(A. Komyo) Center for Mathematical and Data Sciences, Kobe University, 1-1 Rokkodai-cho, Nada-ku, Kobe, 657-8501, Japan
Email address: `akomyo@math.kobe-u.ac.jp`

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Deformation space of circle patterns

Wai Yeung Lam

Abstract

William Thurston proposed regarding the map induced from two circle packings with the same tangency pattern as a discrete holomorphic function. A discrete analogue of the Riemann mapping is deduced from Koebe-Andreev-Thurston theorem. One question is how to extend this theory to Riemann surfaces and relate classical conformal structures to discrete conformal structures. Since circles are preserved under complex projective transformations, we consider circle packings on surfaces with complex projective structures. Kojima, Mizushima and Tan conjectured that for a given combinatorics the deformation space of circle packings is diffeomorphic to the Teichmueller space. In this talk, we report updates on the conjecture and explain the extension to infinite circle patterns on open disks.

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(W. Y. Lam) University of Luxembourg, Department of Mathematics, Maison du nombre, 6 avenue de la Fonte, L-4364 Esch-sur-Alzette, Luxembourg.
Email address: `wai.lam@uni.lu`

The author was partly supported by CoSH, FNR grant O20/14766753.

Moduli space of rank three logarithmic connections on the projective line with three poles

Takafumi Matsumoto

Abstract

An $A_2^{(1)*}$-surface is a smooth projective rational surface in Sakai's classification of Painlevé equations. Arinkin and Borodin noted that moduli space of rank three logarithmic connections on the projective line with three poles is isomorphic to the surface from an $A_2^{(1)*}$-surface minus an anti-canonical divisor from the viewpoint of linear difference equations and the Mellin transform. In this talk, I will explain how to describe the moduli space directly. This method is based on the approach to the Painlevé VI case by Inaba-Iwasaki-Saito. Using this method, we can prove that moduli space of certain parabolic ϕ-connections, which is a compactification of moduli space of logarithmic parabolic connection, is isomorphic to an $A_2^{(1)*}$-surface.

(T. Matsumoto) Departmant of Mathematics, Graduate School of Science, Kobe University, Kobe, Rokko 657-8501, Japan
Email address: `tmatumt@math.kobe-u.ac.jp`

The author was partly supported by JSPS KAKENHI Grant Numbers 22J10695.

Generalizations of the Hermitian-Einstein equation for cyclic Higgs bundles

Natsuo Miyatake

We introduce some generalizations of the Hermitian-Einstein equation for diagonal harmonic metrics on cyclic Higgs bundles, including a generalization using (quasi) subharmonic functions. When the coefficients are all smooth, we prove the existence, uniqueness, and convergence of the solution of their heat equations with Dirichlet boundary conditions. We also generalize two fundamental inequality estimates for solutions of the Hermitian-Einstein equation of cyclic Higgs bundles. This talk is base on the content of the paper [1].

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(N. Miyatake) Institute of Mathematics for Industry, Kyushu University 744 Motooka, Fukuoka 819-0395, Japan
Email address: `n-miyatake@imi.kyushu-u.ac.jp`
Email address: `natsuo.m.math@gmail.com`

Nilpotent Lie algebras obtained by quivers and Ricci solitons

Fumika Mizoguchi

Abstract

Nilpotent Lie groups with left-invariant metrics provide non-trivial examples of Ricci solitons. A quiver is a directed graph where loops and multiple arrows between two vertices are allowed. In this talk, we introduce a new method for obtaining nilpotent Lie algebras from finite quivers without cycles. For all of these Lie algebras, we prove that the corresponding simply connected nilpotent Lie groups admit left-invariant Ricci solitons. This constructs a large family of examples of Ricci soliton nilmanifolds with arbitrarily high nilpotency steps.

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(F. Mizoguchi) Osaka City University, 3-3-138, Sugimoto, Sumiyoshi-ku, Osaka, 558-8585, Japan
Email address: `m21sa032@st.osaka-cu.ac.jp`

This work was partly supported by Osaka Metropolitan University Advanced Mathematical Institute (MEXT Joint Usage/Research Center on Mathematics and Theoretical Physics).

Minimal PF submanifolds in Hilbert spaces with symmetries

Masahiro Morimoto

As a generalization of submanifolds in Euclidean spaces, one can consider submanifolds in Hilbert spaces. In 1988, R. S. Palais and C.-L. Terng [5, 7] introduced a suitable class of submanifolds in Hilbert spaces, which are called *proper Fredholm* (PF) submanifolds. It follows that the shape operators of PF submanifolds are compact self-adjoint operators. Moreover one can apply the infinite dimensional differential topology and Morse theory to PF submanifolds. Palais and Terng [5, 7] gave examples of PF submanifolds which are orbits of the gauge transformations. Later, U. Pinkall and G. Thorbergsson [6] extended those examples and Terng [8] eventually formulated them in the framework of path group actions. Those actions are understood through an equivariant Riemannian submersion called the *parallel transport map* [9] and moreover they have close relations to the infinite dimensional symmetric spaces called *affine Kac-Moody symmetric spaces* [8, 1]. In this talk, I will explain foundations of PF submanifolds in Hilbert spaces, introduce my results concerning minimal PF submanifolds with symmetries [2, 3, 4], and discuss some future works related to integrable systems.

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(M. Morimoto) Osaka Central Advanced Mathematical Institute, Osaka Metropolitan University, 3-3-138, Sugimoto, Sumiyoshi-ku, Osaka, 558-8585, Japan
Email address: `morimoto.mshr@gmail.com`

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Wall-crossing formula for framed quiver moduli

Ryo Ohkawa

Abstract. We study wall-crossing phenomena of moduli of framed quiver representations. These spaces are expected to be very useful to capture representation theoretic nature of special functions for integrable systems. Among these moduli spaces, we have type A flag manifold, type A affine Laumon spaces, Nakajima quiver variety, and framed moduli of sheaves on blow-ups. In particular, we study wall-crossing formulas for integrals of the Euler classes of weighted sums of tautological bundles and tangent bundles of these moduli spaces.

1 Introduction

Framed quiver representation is a quiver representation with a framing. Moduli of framed quiver representations are studied by Reineke [10]. It is expected to be very useful to capture representation theoretic nature of special functions for integrable systems. Among these moduli spaces, we have type A flag manifold, type A affine Laumon spaces, Nakajima quiver variety, and framed moduli of blow-ups.

Mochizuki [3] studied wall-crossing formula for moduli of parabolic sheaves on surfaces. Nakajima-Yoshioka [6] translated it to quiver language from the context of Nekrasov partition functions.

Nekrasov's conjecture states that these partition functions give deformations of the Seiberg-Witten prepotentials for $N = 2$ SUSY Yang-Mills theory. This conjecture is proven in Braverman-Etingof [1], Nekrasov-Okounkov [4] and Nakajima-Yoshioka [5] independently.

On the other hand, another types of wall-crossing formulas are also studied in [7] where walls coincide with hyperplanes perpendicular to imaginary roots. In this paper, we summarize these techniques in general settings of framed quiver moduli.

2 Main results

Let $Q = (Q_0, Q_1, Q_2)$ be a quiver with relations, where Q_0 is the set of vertices, Q_1 is the set of arrows, and Q_2 consists of linear combinations of paths with the same beginning and ending vertices in Q_0. Here path is a composable sequence of arrows. For each arrow $a \in Q_1$, we write by $\mathrm{out}(a)$ and $\mathrm{in}(a)$ the beginning and the ending vertices in Q_0.

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We consider a finite dimensional Q_0-graded vector space $V = \bigoplus_{v \in Q_0} V_v$. A Q-representation B on V is a collection of linear maps

$$B = (B_h) \in \prod_{a \in Q_1} \mathrm{Hom}_{\mathbb{C}}(V_{\mathrm{out}(h)}, V_{\mathrm{in}(h)})$$

satisfying relations defined from Q_2. In this talk, we assume that there exists a vertex $\infty \in Q_0$ such that $\dim V_\infty = 1$. We call ∞ *a framing vertex*, and put $I = Q_0 \setminus \{\infty\}$.

We take stability parameter $\zeta = (\zeta_i)_{i \in I} \in \mathbb{R}^I$ defining stability conditions for Q-representations in a certain way. For the dimension vector $\alpha = (\dim V_v)_{v \in Q_0} \in (\mathbb{Z}_{\geq 0})^{Q_0}$ of V, we write by $M^\zeta(\alpha) = M_Q^\zeta(V)$ moduli of ζ-semistable Q-representations on V.

We take another dimension vector $\beta = (\beta_i)_{i \in I} \in (\mathbb{Z}_{\geq 0})^I$ and a stability parameter $\bar{\zeta}$ from the hyperplane

$$\beta^\perp = \left\{ \zeta = (\zeta_i)_{i \in I} \in \mathbb{R}^I \;\middle|\; \sum_{i \in I} \zeta_i \beta_i = 0 \right\}$$

which is not perpendicular to any other direction of dimension vectors. We have two chambers whose boundaries containing $\bar{\zeta}$, and write by $\mathcal{C}_+$ the one whose element ζ satisfy $(\zeta, \beta) < 0$, and by $\mathcal{C}_-$ the other one.

For $\zeta^\pm \in \mathcal{C}_\pm$, we develop method to compare equivariant integrals over $M^{\zeta^+}(\alpha)$ and $M^{\zeta^-}(\alpha)$, and derive wall-crossing formula among them generalizing the previous works [7], [8] and [9] based on [3], [6]. Furthermore, we apply this formula to the Euler classes of weighted sums of tautological bundles and tangent bundles.

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(R. Ohkawa) Osaka Central Advanced Mathematical Institute, 3-3-138, Sugimoto, Sumiyoshi-ku, Osaka, 558-8585, Japan
Email address: `ohkawa.ryo@omu.ac.jp`

Constructing isometric tori with the same curvatures

Andrew O. Sageman-Furnas

Abstract

Which data determine an immersed surface in Euclidean three-space up to rigid motion? A generic surface is locally determined by only an intrinsic metric and extrinsic mean curvature function. However, there are exceptions. These may arise in a family like the isometric family of vanishing mean curvature surfaces transforming a catenoid into a helicoid, or as a so-called Bonnet pair of surfaces.

For compact surfaces, Lawson and Tribuzy proved in 1981 that a metric and non-constant mean curvature function determine at most one immersion with genus zero, but at most two compact immersions (compact Bonnet pairs) for higher genus.

In this talk, we discuss our recent construction of the first examples of compact Bonnet pairs. It uses a local classification by Kamberov, Pedit, and Pinkall in terms of isothermic surfaces. Moreover, we describe how a structure-preserving discrete theory for isothermic surfaces and Bonnet pairs led to this discovery.

The smooth theory is joint work with Alexander Bobenko (Technical University of Berlin, Germany) and Tim Hoffmann (Technical University of Munich, Germany) and the discrete theory is joint work with Tim Hoffmann and Max Wardetzky (University of Göttingen, Germany).

(A.O. Sageman-Furnas) Department of Mathematics, North Carolina State University, Raleigh, NC 27607, USA.
Email address: `asagema@ncsu.edu`

Some submanifolds of the associative Grassmann manifold

Yuuki Sasaki

A 4-dimensional subspace of the octonions $\mathbb{O}$ satisfying the associative law is called an associative subspace and the associative Grassmann manifold $G_{\mathbb{H}}(\mathbb{O})$ is the set of all associative subspaces. The associative Grassmann manifold is introduced by Harvey-Lawson [3]. The associative Grassmann manifold is a type G irreducible compact symmetric space. Moreover, the associative Grassmann manifold has a quaternionic Kähler structure and becomes a compact quaternionic symmetric space.

In recently, Enoyoshi-Tsukada construct a harmonic map from a Lagrangian submanifold of S^6 from $G_{\mathbb{H}}(\mathbb{O})$ [2]. In this talk, we introduce a construction of a totally complex harmonic immersion from an almost complex submanifold of S^6 into $G_{\mathbb{H}}(\mathbb{O})$ by using a Gauss map. Using similar arguments, we introduce a construcion of a CR immersion from a 3-dimensional CR submanifold of S^6 into $G_{\mathbb{H}}(\mathbb{O})$.

Moreover, if time permits, we will introduce the following results. Enoyoshi-Tsukada prove that a polar of $G_{\mathbb{H}}(\mathbb{O})$ is an image of a totally complex immersion, where a polar is a connected component of the fixed point set of a geodesic symmetry [1]. It is known that a polar is obtained as an orbit of the action of the isotropy group of the isometry group. In this talk, we observe some properties of each orbit of the isotropy group action with respect to the quaternionic Kähler structure. In particular, we observe that many orbits are an image of a CR immersion which has similar properties of a totally complex immersion.

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(Yuuki Sasaki) National Institute of Technology, Tokyo College, 1220-2, Kunugida-machi, Hachioji-shi, Tokyo, 193-0997, Japan
Email address: `y_sasaki@tokyo.kosen-ac.jp`

Integrable systems in four independent variables related to contact structures in dimension three

Artur Sergyeyev

We start with a review of our construction [1] (cf. also [2]) that gave rise to a large new class of nonlinear partial differential systems in four independent variables integrable in the sense of soliton theory using a certain contact structure in dimension three. We then proceed to explore the possibility of extending this construction to more general contact structures in the same dimension.

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(A. Sergyeyev) Mathematical Institute, Silesian University in Opava, Na Rybníčku 1, 74601 Opava, Czech Republic

Email address: `artur.sergyeyev@math.slu.cz`

On a heat equation of $SL(2, \mathbb{R})$, and a future direction of its q-deformation

Masafumi Shimada

Abstract

In this talk, we discuss several formulae of the heat kernel on $SL(2, \mathbb{R})$, and a future application of these formulae to a q-deformed heat equation on a quantum group. In previous studies on the heat kernel in a unimodular Lie group of type I, including the case of $SL(2, \mathbb{R})$, we use a noncommutative-harmonic-analytic technique, called generalized noncommutative Fourier transform. Our formula is, with the help of weighted Maass Laplacians, another explicit form of the heat kernel on $SL(2, \mathbb{R})$, which explains an intrinsic structure of the heat kernel in terms of hyperbolic geometry.

We recall the fact that, given a parabolic partial differential equation on a Riemannian manifold, its heat kernel plays an important role in its initial value problem. Motivated by this fact, we hopefully apply a q-deformation of our heat kernel formula to a study on a q-difference heat equation of a quantum group $SL_q(2)$ in the context of quantum Riemannian geometry.

(M. Shimada) Faculty of Mathematics, Kyushu University, 744, Motooka, Nishi-ku, Fukuoka-shi,Fukuoka, 819-0395, Japan
Email address: `m.shimada.a90@s.kyushu-u.ac.jp`

Geometry of anisotropic double crystals and classification of some examples

Eriko Shinakwa, Miyuki Koiso
(Presenter: Eriko Shinkawa)

There was a long-standing conjecture which was called the double bubble conjecture. It says that the standard double bubble provides the least-perimeter way to enclose and separate two given volumes, here the standard double bubble is consisting of three spherical caps meeting along a common circle at 120-degree angles. This conjecture had been believed since about 1870 and was proved in 2002 by M. Hutchings et al. in $\mathbb{R}^3$, and a student of F. Morgan extended it to higher dimensions.

Double bubbles are a mathematical model of soap bubbles. The energy functional is the total area of the surface. On the other hand, when we think about a mathematical model of anisotropic substance like crystals, we need to consider the energy density function $\gamma : S^n \to \mathbb{R}^+$ depending on the normal direction ν of the surface, where $S^n = \{X \in \mathbb{R}^{n+1} \mid |X| = 1\}$ is the n-dimensional unit sphere in $\mathbb{R}^{n+1}$. γ is called an anisotropic energy density function, and its sum (integral) over the surface is called an anisotropic energy. The surface is a constant anisotropic mean curvature (CAMC) surface if it is a critical point of the anisotropic energy for all volume-preserving variations. CAMC surfaces are a generalization of constant mean curvature surfaces.

In this study, we extend the double bubble problem to a double crystal (DC) problem, that is, we minimize the anisotropic energy instead of the surface area. The solutions are a mathematical model of multiple crystals. There were some previous studies relating to the DC problem. G. R. Lawlor ([1]) determined the energy-minimizer for the case where each energy density function γ_i $(i = 0, 1, 2)$ is constant. For $n = 1$, F. Morgan et al. studied the shapes of the minimizers of the total anisotropic energy of curves among curves enclosing prescribed areas. Especially, they determined the shapes of all minimizers for the case of $\gamma_i\,(\nu_1, \nu_2) = |\nu_1| + |\nu_2|$ $(i = 0, 1, 2)$ $((\nu_1, \nu_2) \in S^1)$ ([2]).

There is a unique hypersurface that minimizes the anisotropic energy among all closed hypersurfaces enclosing the same volume, and this hypersurface is known as the Wulff shape. In this study, we assume that the Wulff shape is smooth. We will derive the first variation formula for the anisotropic energy and obtain the conditions for a hypersurface to be a double crystal ([3], [4]). In particular, for $n = 1$, for a kind of special energy density functions, some of which approximate the energy studied in [2], we classify the double crystals in terms of symmetry and the given areas.

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(E. Shinkawa) Advanced Institute for Materials Research, Tohoku University, 2-1-1 Katahira, Aoba-ku, Sendai 980-8577, Japan
Email address: `eriko.shinkawa.e8@tohoku.ac.jp`

(M. Koiso) Institute of Mathematics for Industry, Kyushu University, 744 Motooka, Nishi-ku, Fukuoka 819-0395, Japan
Email address: `koiso@imi.kyushu-u.ac.jp`

Smooth and discrete constrained elastic curves in space forms

Gudrun Szewieczek

Euler's famous *elastic curves* in the Euclidean plane are obtained as the critical points of the bending energy while fixing the length of the curve. If, additionally, also the sector area is constrained, then the curve is called *constrained elastic*. Alternative characterizations in terms of the curvature [6] or the existence of a special 'axis' [1, 4] allow for natural generalizations to space forms.

Recent interest in constrained elastic curves in space forms is caused by the fact that those arise as planar or spherical curvature lines of many integrable surface classes: for example, on constrained Willmore tori [3, 5], isothermic surfaces with planar curvature lines [2] and, more generally, on Lie applicable surfaces [4].

In this talk we shall give an overview on smooth constrained elastic curves in space forms and discuss an integrable discretization. The latter is joint work with Tim Hoffmann and Jannik Steinmeier.

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(G. Szewieczek) Technical University of Berlin, Institute of Mathematics MA 8-4, 10623 Berlin, Germany
Email address: `szewieczek@math.tu-berlin.de`

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A maximal element of a moduli space of Riemannian metrics

Yuichiro Taketomi

1 Introduction

Let X be a connected smooth manifold, and denote by $\mathfrak{M}(X)$ the set of all smooth Riemannian metrics on X. Define a equivalent relation $\sim$ on $\mathfrak{M}(X)$ as follows:

$$g \sim h \;:\Leftrightarrow\; \begin{array}{l}\text{There exists some } c > 0 \text{ such that two Riemannian manifolds} \\ (X, cg) \text{ and } (X, h) \text{ are isometric with each other.}\end{array}$$

Denote by $[g]$ the equivalent class of $g \in \mathfrak{M}(X)$ with respect to $\sim$. Now define a preorder $\prec$ on the moduli space $\mathfrak{M}(X)/_\sim$ as follows:

$$[g] \prec [h] \;:\Leftrightarrow\; \begin{array}{l}\text{For all } g' \in [g] \text{ there exists some } h' \in [h] \text{ such that} \\ \mathrm{Isom}(X, g') \subset \mathrm{Isom}(X, h').\end{array}$$

Note that the preorder $\prec$ is not a partial order in general. Namely, the order $\prec$ satisfies reflexivity and transitivity, but does not satisfy asymmetry.

Definition (T.). *A Riemannian metric $g \in \mathfrak{M}(X)$ is said to be maximal if the equivalent class $[g]$ is a maximal element of the preordered space $(\mathfrak{M}(X)/_\sim, \prec)$ (i.e. $[g] \prec [h]$ implies $[g] = [h]$).*

One can see that a Riemannian metric $g \in \mathfrak{M}(X)$ is maximal if and only if

$$\mathrm{Isom}(X, g) \subset \mathrm{Isom}(X, h) \;\Rightarrow\; [g] = [h]. \tag{$\sharp$}$$

Now we consider a metric evolution equation $\partial_t g_t = \mathcal{R}(g_t)$ on X. Here, $\mathcal{R}$ is a map from $\mathfrak{M}(X)$ to the set of all symmetric $(0,2)$-tensor on X. A typical example is a Ricci flow (*i.e.* $\mathcal{R} = -2\mathrm{Ric}$). A solution $\{g_t\}_{t\in[0,T)}$ of a metric evolution equation is said to be *self-similar* if $[g_0] = [g_t]$ for all $t \in [0, T)$. By the property ($\sharp$), one has

Proposition. *Let $\partial_t g_t = \mathcal{R}(g_t)$ be a metric evolution equation, and $\{g_t\}_{t\in[0,T)}$ be a solution of the equation whose initial metric g_0 is a maximal metric. If the solution $\{g_t\}_{t\in[0,T)}$ preserves the isometry group in the sense that $\mathrm{Isom}(X, g_0) \subset \mathrm{Isom}(X, g_t)$ for all $t \in [0, T)$, then the solution $\{g_t\}_{t\in[0,T)}$ is self-similar.*

It is known that a solution of the Ricci flow whose initial metric is complete and has bounded curvature exists uniquely ([1, 3]), and hence preserves the isometry group. Hence a complete maximal metric with bounded curvature (in particular, a homogeneous maximal metric) is a Ricci soliton. Maximal metrics give examples of self-similar solutions not only for the Ricci flow but also for any geometric evolution equations which preserve isometry groups.

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2 Examples of maximal metrics

Recall that a Riemannian manifold (X, g) is called an *isotropy irreducible* if the isotropy representation $\mathrm{Isom}(X, g)_p \curvearrowright T_pX$ is an irreducible representation for each $p \in X$. Many geometer have studied the classification problem of isotropy irreducible spaces (*e.g.* [7, 8]), and in the end, complete connected isotropy irreducible spaces have been classified by Wang-Ziller ([5]). By the Schur's lemma, one has

Proposition. *For a complete connected isotropy irreducible space (X, g), the metric g is a maximal metric.*

Conversely, for the compact case, maximal metrics are isotropy irreducible metrics:

Theorem (T.). *Let (X, g) be a compact connected Riemannian manifold. If g is maximal then (X, g) is isotropy irreducible.*

On the other hand, one can show that there are many examples of maximal metrics on the noncompact manifold $\mathbb{R}^k$. For examples,

Proposition (T.). *For $W = (w_2, w_3, \ldots, w_n) \in \mathbb{R}^{n-1}$, the following Riemannian metric $g_W \in \mathfrak{M}(\mathbb{R}^n)$ is a maximal metric:*

$$g_W := (dx_1)^2 + e^{-2w_2x_1}(dx_2)^2 + \cdots + e^{-2w_nx_1}(dx_n)^2.$$

The metric g_W is isotropy irreducible if and only if $w_2 = w_3 = \cdots = w_n$.

Note that the Riemannian metric g_W is isometric to some left-invariant metric on some simply connected solvable Lie group. Two metrics $g_W, g_{W'} \in \mathfrak{M}(\mathbb{R}^n)$ are isometric if and only if the two subspaces $W, W' \in \mathbb{R}^{n-1}$ coincide under some permutation $\sigma \in \mathfrak{S}_{n-1}$. Hence the family $\{g_W\}_{W \in \mathbb{R}^{n-1}}$ gives a continuous family of maximal metrics.

Let $\mathcal{G}$ be a simple finite graph with p vertices and q edges. Denote by $\mathcal{V} := \{v_1, v_2, \ldots, v_p\}$ and $\mathcal{E} := \{e_{p+1}, e_{p+2}, \ldots, e_{p+q}\}$ the set of (numbered) vertices and the set of (numbered) edges of $\mathcal{G}$, respectively. For $j \in \{p+1, \ldots, p+q\}$, define $s_j, t_j \in \{1, 2, \ldots, p\}$ as follows:

two vertices v_{s_j} and v_{t_j} are jointed by the edge e_j, where $s_j < t_j$.

Then we define a Riemannian metric $h_\mathcal{G} \in \mathfrak{M}(\mathbb{R}^{p+q})$ as follows:

$$h_\mathcal{G} = \sum_{i \leq p} (dx_i)^2 + \sum_{j \geq p+1} (dx_j + \frac{1}{2} x_{t_j} dx_{s_j} - \frac{1}{2} x_{s_j} dx_{t_j})^2.$$

Note that the Riemannian metric $h_\mathcal{G}$ is isometric to some left-invariant metric on some simply connected nilpotent Lie group ([2]).

Proposition (T.). *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a simple finite edge-transitive graph with the vertex set $\mathcal{V} = \{v_1, v_2, \ldots, v_p\}$ and the edge set $\mathcal{E} = \{e_{p+1}, e_{p+2}, \ldots, e_{p+q}\}$. Then the metric $h_\mathcal{G} \in \mathfrak{M}(\mathbb{R}^{p+q})$ is a maximal metric which is not isotropy irreducible.*

Here, a simple graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is called an *edge-transitive graph* if the automorphism group $\mathrm{Aut}(\mathcal{G})$ acts on the edge set $\mathcal{E}$ transitively. Note that two metrics $h_\mathcal{G}$ and $h_{\mathcal{G}'}$ are isometric if and only if the two graphs $\mathcal{G}$ and $\mathcal{G}'$ are isomorphic as graph ([2, 6]). Hence the family $\{h_\mathcal{G}\}_\mathcal{G}$ gives infinitely many examples of maximal metrics.

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(Y. Taketomi) Osaka Central Advanced Mathematical Institute, 3-3-138, Sugimoto, Sumiyoshi-ku, Osaka, 558-8585, Japan

Email address: `v21255l@omu.ac.jp`

Immersed special Lagrangians and mean curvature flow

Albert Wood

Abstract

PDE gluing constructions have become an indispensable tool in modern geometric analysis. To give just a couple of examples, Joyce utilised gluing constructions both to provide the first examples of compact G2 manifolds and to desingularise special Lagrangians with conical singularities. Recent pioneering work of Brendle-Kapouleas describes a parabolic gluing construction for Ricci flow, giving an example of an ancient flow beginning at a singular torus quotient.

This talk is based on upcoming joint work with Chung-Jun Tsai and Wei-Bo Su, in which we develop a parabolic gluing construction for Lagrangian submanifolds in Calabi-Yau manifolds, inspired by the work of Joyce and Brendle-Kapouleas. Explicitly, we will demonstrate that any special Lagrangian submanifold in a Calabi-Yau manifold $L \subset M$ with a single immersed point $x \in L$ is the infinite-time singular limit of a mean curvature flow of Lagrangian submanifolds $F_t : \widetilde{L} \to M$, $t \in [\Lambda, \infty)$.

(A. Wood) Hsiu-Chi House, No 16-1 Siyuan Street, Zhongzheng District, Taipei City 100, Taiwan
Email address: `awood@ncts.ntu.edu.tw`
webpage: `www.albert-maths.co.uk`

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